C^*-Completions of Hecke algebras and crossed products by Hecke pairs

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Environmental issues

The carbon emissions due to research related travelling during the three and a half years of this Ph.D. project ammounted to almost 5 metric tons of CO_2. Travelling seems to be, by far, the part of this work with the largest carbon footprint (the office electricity usage, for example, could be roughly estimated to account for an emission of 0.03 metric tons of CO_2).

Regrettably, the author has not done anything to offset this record...

(estimates made through the website: http://www.carbonfootprint.com)

To my parents,
João and Carmen

Contents

Introduction

The study of C^*-algebras associated with groups and dynamical systems is a central topic in the field of operator algebras, dating back to the foundational works of Murray and von Neumann in this area. On one side, groups and dynamical systems give rise to a rich source of examples of C^*-algebras with interesting properties, and conversely the study of these C^*-algebras provides valuable information about the original groups or dynamical systems. This interplay between C^*-algebras and groups and dynamical systems reflects how the field of operator algebras has been intertwined with other areas of mathematics such as harmonic analysis, representation theory of groups, topological dynamics, and more recently, number theory.

The relevant C^*-algebras associated to a given group arise as C^*-completions of its group algebra, and there are two such C^*-completions which are canonical: the full group C^*-algebra and the reduced group C^*-algebra. For a dynamical system, by which we roughly mean a group action on a *-algebra, one typically studies its associated C^*-crossed products, which are C^*-algebras that encode information about the dynamics. There are two canonical C^*-crossed products, a full and a reduced one, and group C^*-algebras are a particular case of crossed products (arising from a trivial group action).

Given a normal subgroup Γ of a group G, the quotient G/Γ is also a group and therefore we can talk about its group algebra and about dynamical systems involving actions of G/Γ. Normality of the subgroup Γ may seem to be a natural requirement, as otherwise the quotient G/Γ does not have a canonical group structure. It is however still possible to give some meaning to the group algebra of the "quotient" G/Γ for subgroups Γ which are "almost normal" in G, even though G/Γ is not necessarily a group. This is the basic idea behind Hecke pairs and Hecke algebras.

A *Hecke pair* (G, Γ) consists of a group G and a subgroup $\Gamma \subseteq G$ for which every double coset $\Gamma g \Gamma$ is the union of finitely many left cosets. In this case Γ is also said to be a *Hecke subgroup* of G. Examples of Hecke subgroups include finite subgroups, finite-index subgroups and normal subgroups. Hecke subgroups are also sometimes called *almost normal subgroups* (although we will not use this terminology here) and it is in fact many times insightful to think of this definition as a generalization of the notion of normality of a subgroup. The *Hecke algebra* $\mathcal{H}(G,\Gamma)$ of a Hecke pair (G,Γ) is a *-algebra of functions over the set of double cosets $\Gamma\backslash G/\Gamma$, with a suitable convolution product and involution. It generalizes the definition of the group algebra $\mathbb{C}(G/\Gamma)$ of the quotient group when Γ is a normal subgroup.

Although Hecke algebras retain some similarities with group algebras, they are also quite different in many aspects. The most basic and striking difference is the fact that, unlike group algebras, the canonical basis of a Hecke algebra does not consist solely of unitary elements, in general. Another important point of distinction, and one which is of special relevance in this work, has to do with their representation theory. As it is well-known, there is a bijective correspondence between nondegenerate *-representations of a group algebra and unitary representations of the underlying group. Hall [18] asked if something analogous was true for Hecke pairs, i.e. if there is a bijective correspondence between nondegenerate *-representations of a Hecke algebra $\mathcal{H}(G,\Gamma)$ and unitary representations of G generated by the Γ-fixed vectors. Hall showed in [18] that this is not the case in general. However, this correspondence has been established for several classes of Hecke pairs. Whenever this correspondence holds we say that the Hecke pair satisfies *Hall's equivalence.*

Main goals

The present work has two main goals. The first goal is to study C^*-completions of Hecke algebras and understand when certain of these completions coincide, and to use this information to establish Hall's equivalence for several classes of Hecke pairs. The second goal is to develop a theory of crossed products by Hecke pairs with a view towards applications in non-abelian duality. This thesis is therefore divided into two parts, accordingly, which are not completely disjoint, there being a direct influence of the first part on the second.

Part I

The interest in Hecke algebras in the realm of operator algebras was to a large extent raised through the work of Bost and Connes [6] on phase transitions in number theory, and since then several authors have studied C^*-algebras which arise as completions of Hecke algebras. There are several canonical C^*-completions of a Hecke algebra $\mathcal{H}(G,\Gamma)$ which one can consider and these are: $C^*(G,\Gamma)$, $C^*(L^1(G,\Gamma))$, $pC^*(\overline{G})p$ and $C^*_r(G,\Gamma)$ (the reader is referred to [42] and [24], or the first chapter of this thesis, for the detailed definitions).

The full Hecke C^*-algebra $C^*(G,\Gamma)$ is the maximal C^*-completion, i.e. the enveloping C^*-algebra, of the Hecke algebra $\mathcal{H}(G,\Gamma)$. In clear contrast to the case of group algebras of discrete groups, $C^*(G,\Gamma)$ does not always exist. The question of existence of the full Hecke C^*-algebra has been of particular interest ([6], [2], [7], [18], [42], [16], [29], [5], [24], [12]). One of the reasons for that, firstly explored by Hall [18], has to do with Hall's equivalence. It was shown in [18] that for Hall's equivalence to hold it is necessary that the Hecke algebra has an enveloping C^*-algebra, which does not always happen. It was later clarified by Kaliszewski, Landstad and Quigg [24] that Hall's equivalence holds for a Hecke pair (G,Γ) precisely when the enveloping C^*-algebra $C^*(G,\Gamma)$ exists and one has $C^*(G,\Gamma)\cong C^*(L^1(G,\Gamma))\cong pC^*(\overline{G})p$.

The problem of deciding if a Hecke algebra has an enveloping C^*-algebra seems to be of a non-trivial nature, with satisfactory answers, arising from various distinct methods, known only for certain classes of Hecke pairs.

One of the main motivations for the present work is to give a unified approach to this problem for a large class of Hecke pairs. We recover most of the known cases in the literature but we also obtain several new ones. We achieve this in Chapter 2 by associating a directed graph to a Hecke algebra $\mathcal{H}(G,\Gamma)$, whose vertices are the double cosets and whose directed edges are determined by how products of the form $(\Gamma g\Gamma)^* * \Gamma g\Gamma$ decompose as sums of double cosets. We prove that finiteness of the *co-hereditary set* generated by a vertex $\Gamma g\Gamma$, i.e. the set of vertices one encounters by moving forward in the graph starting from $\Gamma g\Gamma$, implies that

$$\sup_{\pi}\|\pi(\Gamma g\Gamma)\|<\infty\,,$$

where the supremum runs over the *-representations of the Hecke algebra. Thus, analysing these co-hereditary sets gives valuable information regarding the existence of enveloping C^*-algebras. Moreover, we prove that if *all* double cosets generate finite co-hereditary sets, then the full Hecke C^*-algebra $C^*(G,\Gamma)$ exists and coincides with $C^*(L^1(G,\Gamma))$.

We develop certain tools, based on iterated commutators in the group G, that allow us to show that our assumptions hold in a variety of classes of Hecke pairs, and thus enable us to answer affirmatively the question of existence of enveloping C^*-algebras of the corresponding Hecke algebras. Some of the new results we prove state that if a group G satisfies some generalized nilpotency property, then for *any* Hecke subgroup Γ the Hecke algebra

$\mathcal{H}(G,\Gamma)$ has an enveloping C^*-algebra which coincides with $C^*(L^1(G,\Gamma))$.

We point out that our results show that the classes of Hecke algebras studied in the present work, and therefore many of those studied in the literature, satisfy a stronger property than just having an enveloping C^*-algebra: they are in fact BG^*-algebras. The standard reference for this class of *-algebras is Palmer [35], but we also give a short description in Chapter 1. The reason for considering this stronger property is not only because of how well-behaved these *-algebras are, but also because of their relevance in the study of crossed products by Hecke pairs, developed in the second part of this work.

The problem of deciding for which Hecke pairs one has the isomorphism $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ is also only partially understood. It is known that this isomorphism holds for the class of Hecke pairs which satisfy Hall's equivalence, since for this class we have more generally $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$. Also, it was proven by Kaliszewski, Landstad and Quigg [24] that $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ whenever the Schlichting completion $\overline{G}$ is a Hermitian group.

Another motivation for the present work comes from the wish to establish the isomorphism $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ for the class of groups G of subexponential growth. We achieve this in Chapter 3 by proving a more general result, and obtaining also Kaliszewski, Landstad and Quigg's result for Hermitian groups as a corollary. For that we introduce the notion of a *quasi-symmetric* group algebra: a locally compact group G will be said to have a quasi-symmetric group algebra if for any $f \in C_c(G)$ the spectrum of $f^* * f$ relative to $L^1(G)$ is in $\mathbb{R}_0^+$. It follows directly from the definition that Hermitian groups have a quasi-symmetric group algebra and it follows from the work of Hulanicki ([22], [21]) that this is also the case for groups of subexponential growth. We show that $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ whenever the Schlichting completion $\overline{G}$ has a quasi-symmetric group algebra.

By combining our result on quasi-symmetric group algebras, with our results on the full Hecke C^*-algebra, and also a Theorem of Tzanev [42] concerning $C_r^*(G,\Gamma)$, we are able to establish in Section 3.4 that $C^*(G,\Gamma)$ exists and $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$ for several classes of Hecke pairs, including all Hecke pairs (G,Γ) where G is a nilpotent group. Consequently, it follows that Hall's equivalence holds for all such classes of Hecke pairs.

It is natural to ask if the various canonical C^*-completions of Hecke algebras are in general different, or if they coincide for all Hecke pairs. In other words, are there examples of Hecke pairs for which we have $C^*(G,\Gamma) \ncong C^*(L^1(G,\Gamma))$ or $C^*(L^1(G,\Gamma)) \ncong pC^*(\overline{G})p$? This question was already raised in [24] and in this setting we observe that the case $pC^*(\overline{G})p \ncong C_r^*(G,\Gamma)$ is well understood to be a matter of non-amenability, following Tzanev [42].

According to [24], Tzanev claims in private communication with Kaliszewski, Landstad and Quigg that the Hecke pair $(PSL_3(\mathbb{Q}_q), PSL_3(\mathbb{Z}_q))$ is such that $C^*(L^1(G,\Gamma)) \ncong pC^*(\overline{G})p$, but no proof has been published and no other example seems to be known, as far as we know. We prove in Section 3.5 that $C^*(L^1(G,\Gamma)) \ncong pC^*(\overline{G})p$ for the Hecke pair $(PSL_2(\mathbb{Q}_q), PSL_2(\mathbb{Z}_q))$, as suggested by Kaliszewski, Landstad and Quigg in [24], but following a different approach than the one they suggest which does not use the representation theory of $PSL_2(\mathbb{Q}_q)$.

Part II

Heuristically, a crossed product of an algebra A by a Hecke pair (G,Γ) should be thought of as a crossed product (in the usual sense) of A by an "action" of G/Γ. The quest for a sound definition of crossed products by Hecke pairs may seem hopelessly flawed since G/Γ is not necessarily a group and thus it is unclear how it should "act" on A. Nevertheless, we will see that in some circumstances such a definition can be given in a meaningful way, recovering the original one whenever G/Γ is a group.

The term "crossed product by a Hecke pair" was first used by Tzanev [41] in order to give another perspective on the work of Connes and Marcolli [8]. This point of view was later formalized by Laca, Larsen and Neshveyev in [30], where they defined a C^*-algebra which can be interpreted as a reduced C^*-crossed product of a commutative C^*-algebra by a Hecke pair.

It seems to be a very difficult task to define crossed products of *any* given algebra A by a Hecke pair, and for this reason we set as our goal to define a crossed product by a Hecke pair in a generality that will cover the following aspects:

- existence of a canonical spanning set of elements in the crossed product;
- possibility of defining covariant representations;
- the Hecke algebra must be a trivial example of a crossed product by a Hecke pair;
- the classical definition of a crossed product must be recovered whenever G/Γ is a group;
- our construction should agree with that of Laca, Larsen and Neshveyev, whenever they are both definable;

- our definition should be suitable for applications in non-abelian duality.

We develop in Chapter 6 a theory of crossed products of certain algebras A by Hecke pairs which takes into account the above requirements. Our approach makes sense when A is an algebra of sections of a Fell bundle over a discrete groupoid. To summarize our set up: we start with a Hecke pair (G, Γ), a discrete groupoid X where G acts, a Fell bundle $\mathcal{A}$ over X with the same fiber over every element in the same G-orbit, and we assume that the G-action on X satisfies some "nice" properties. From this we give the orbit space X/Γ a groupoid structure and we obtain a new Fell bundle $\mathcal{A}/\Gamma$ over this groupoid. We can then define a *-algebra

$$C_c(\mathcal{A}/\Gamma) \times^{alg} G/\Gamma\,,$$

which can be thought of as the crossed product of $C_c(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ). We should point out that a *crossed product* for us is simply a *-algebra, which we can then complete with different C^*-norms or an L^1-norm. Hence, we will always use the symbol $\times^{alg}$ when talking about the (uncompleted) *-algebraic crossed product.

Our construction gives back the usual crossed product construction when Γ is a normal subgroup of G. Moreover, many of the features present in crossed products by discrete groups carry over to our setting. For instance, the role of the group G/Γ is played by the Hecke algebra $\mathcal{H}(G, \Gamma)$, which embeds in a natural way in the multiplier algebra of $C_c(\mathcal{A}/\Gamma) \times^{alg} G/\Gamma$.

The representation theory of crossed products by Hecke pairs (developed in Section 6.2) has many similarities with the group case, but some distinctive new features arise. For instance, as it is well-known in the group case, there is a bijective correspondence between nondegenerate representations of a crossed product $A \times G$ and the so-called covariant representations of A and G, which are certain pairs of unitary representations of G and representations of A. We will show that something completely analogous occurs for Hecke pairs, but in this case one is obliged to consider *pre-representations* of the Hecke algebra, i.e. representations of $\mathcal{H}(G, \Gamma)$ as (possibly) unbounded operators. This consideration was unnecessary in the group case because unitary operators are automatically bounded.

Similarly to crossed products by groups, one can make sense of regular representations in the Hecke pair case by using the regular representation of the Hecke algebra on $\ell^2(G/\Gamma)$. From regular representations it is then possible to define *reduced* C^*-crossed products. Since the algebra $C_c(\mathcal{A}/\Gamma)$ admits several C^*-completions there are several reduced C^*-crossed products that one can form, as for example $C_r^*(\mathcal{A}/\Gamma) \times_r G/\Gamma$ and $C^*(\mathcal{A}) \times_r G/\Gamma$, each of these thought of as the reduced C^*-crossed product of $C_r^*(\mathcal{A}/\Gamma)$, respectively

$C^*(\mathcal{A}/\Gamma)$, by the Hecke pair (G,Γ). These reduced C^*-crossed products have always a faithful conditional expectation onto $C^*_r(\mathcal{A}/\Gamma)$ (respectively, $C^*(\mathcal{A}/\Gamma)$), a property that determines the reduced crossed product uniquely, just like reduced crossed products by groups. This is achieved in Chapters 7 and 8.

Complementing the reduced setting, one would like to form different *full* C^*-crossed products, as $C^*_r(\mathcal{A}/\Gamma)\times G/\Gamma$ and $C^*(\mathcal{A}/\Gamma)\times G/\Gamma$, but in general their existence is not assured. They will always exist, however, if the Hecke algebra is a BG^*-algebra, which as we establish in the first part of this thesis, is a property that is satisfied by several classes of Hecke pairs, including most of those studied in the literature for which $C^*(G,\Gamma)$ is known to exist. Full C^*-crossed products are studied in detail in Chapter 9.

This theory of crossed products by Hecke pairs, which uses Fell bundles over groupoids, is intended for applications in non-abelian duality theory. We develop completely a Stone-von Neumann type theorem for Hecke pairs which encompasses the work of an Huef, Kaliszewski and Raeburn [19], and we envisage for future work a form of Katayama duality with respect to Echterhoff-Quigg's "crossed product" [10].

The Stone-von Neumann theorem, in the language of crossed products by groups, states that for the action of translation of G on $C_0(G)$ we have

$$C_0(G)\times G \;\cong\; C_0(G)\times_r G \;\cong\; \mathcal{K}(\ell^2(G))\,.$$

In [19] an Huef, Kaliszweski and Raeburn introduced the notion of *covariant pairs* of representations of $C_0(G/\Gamma)$ and $\mathcal{H}(G,\Gamma)$, for a Hecke pair (G,Γ), and proved that all covariant pairs are amplifications of a certain "regular" covariant pair. Their result was proven without using or defining crossed products, and can also be thought of as a Stone-von Neumann theorem for Hecke pairs. Using our construction we express their result in the language of crossed products (Chapter 10). In fact, it can be shown that the full crossed product $C_0(G/\Gamma)\times G/\Gamma$ always exists and one has

$$C_0(G/\Gamma)\times G/\Gamma \;\cong\; C_0(G/\Gamma)\times_r G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma))\,.$$

Moreover, our notion of a covariant representation coincides with the notion of a covariant pair of [19], and an Huef, Kaliszewski and Raeburn's result follows as a direct corollary of the above isomorphisms.

Our construction was very much influenced and developed with the wish of obtaining a form of Katayama duality for homogeneous spaces (those arising from Hecke pairs). Even though this has been left for future work, we shall nevertheless explain what we have in mind and how our set up is suitable for tackling this problem, also as a justification of the generality of our assumptions in the definition of crossed products.

Katayama's duality theorem [25] is an analogue for coactions of the duality theorem of Imai and Takai. One version of it states the following: given a coaction δ of a group G on a C^*-algebra A and denoting by $A \times_\delta G$ the corresponding crossed product, we have a canonical isomorphism $A \times_\delta G \times_{\widehat{\delta},\omega} G \cong A \otimes \mathcal{K}(\ell^2(G))$, for some crossed product by the dual action $\widehat{\delta}$ of G. We would like to extend this result to homogeneous spaces coming from Hecke pairs. In spirit we hope to obtain an isomorphism of the type:

$$A \times_\delta G/\Gamma \times_{\widehat{\delta},\omega} G/\Gamma \cong A \otimes \mathcal{K}(\ell^2(G/\Gamma)) \,.$$

The C^*-algebra $A \times_\delta G/\Gamma$ should be a crossed product by a coaction of the homogeneous space G/Γ, while the second crossed product should be by the "dual action" of the Hecke pair (G, Γ) in our sense. It does not make sense in general for a homogeneous space to coact on a C^*-algebra, but it is many times possible to define C^*-algebras which can be thought of as crossed products by coactions of homogeneous spaces ([9], [10]).

It is our point of view that $A \times_\delta G/\Gamma$ should be a certain C^*-completion of the *-algebra $C_c(\mathcal{A} \times G/\Gamma)$ defined by Echterhoff and Quigg [10], which we dub Echterhoff and Quigg's crossed product (a terminology used in [19] for $C^*(\mathcal{A} \times G/\Gamma)$ in case of a maximal coaction). We explain in Chapter 11 how our set up for defining crossed products by Hecke pairs is suitable for achieving such a Katayama duality result for Echterhoff and Quigg's crossed product, and bring insight into the emerging theory of crossed products by coactions of homogeneous spaces.

Part I

C^*-Completions of Hecke Algebras

Chapter 1

Preliminaries

In this chapter we set up the conventions, notation, and background results which will be used throughout this thesis, or at least throughout this first part of the thesis. The topics covered include: *-algebras and their (pre-)*-representations; Hecke algebras, their C^*-completions and representation theory; directed graphs; groups of subexponential growth. We indicate the references where the reader can find more details, but we also provide proofs for those results which we could not find in the literature.

Convention. *The following convention for displayed equations will be used throughout this thesis: if a displayed formula starts with the equality sign, it should be read as a continuation of the previously displayed formula.*

A typical example takes the following form:

$$\begin{aligned} (\text{expression } 1) &= (\text{expression } 2) \\ &= (\text{expression } 3)\,. \end{aligned}$$

By Theorem A and Lemma B it then follows that

$$\begin{aligned} &= (\text{expression } 4) \\ &= (\text{expression } 5)\,. \end{aligned}$$

Under our convention starting with the equality sign in the second array of equations simply means that (expression 3) *is equal to* (expression 4)*.*

1.1 *-Algebras and (pre-)*-representations

Let $\mathscr{V}$ be an inner product space over $\mathbb{C}$. Recall that a function $T : \mathscr{V} \to \mathscr{V}$ is said to be *adjointable* if there exists a function $T^* : \mathscr{V} \to \mathscr{V}$ such that

$$\langle T\xi \, , \, \eta \rangle = \langle \xi \, , \, T^*\eta \rangle \, ,$$

for all $\xi, \eta \in \mathscr{V}$. Recall also that every adjointable operator T is necessarily linear and that T^* is unique and adjointable with $T^{**} = T$. We will use the following notation:

- $L(\mathscr{V})$ denotes the *-algebra of all adjointable operators in $\mathscr{V}$
- $B(\mathscr{V})$ denotes the *-algebra of all bounded adjointable operators in $\mathscr{V}$.

Of course, we always have $B(\mathscr{V}) \subseteq L(\mathscr{V})$, with both *-algebras coinciding when $\mathscr{V}$ is a Hilbert space (see, for example, [35, Proposition 9.1.11]).

Following [35, Def. 9.2.1], we define a *pre-*-representation* of a *-algebra A on an inner product space $\mathscr{V}$ to be a *-homomorphism $\pi : A \to L(\mathscr{V})$ and a **-representation* of A on a Hilbert space $\mathscr{H}$ to be a *-homomorphism $\pi : A \to B(\mathscr{H})$. As in [34, Def. 4.2.1], a pre-*-representation $\pi : A \to L(\mathscr{V})$ is said to be *normed* if $\pi(A) \subseteq B(\mathscr{V})$, i.e. if $\pi(a)$ is a bounded operator for all $a \in A$. We now make a seemingly similar definition, but where the focus is on the elements of the *-algebra, instead of its pre-*-representations:

Definition 1.1.1. Let A be a *-algebra. An element $a \in A$ is said to be *automatically bounded* if $\pi(a) \in B(\mathscr{V})$ for any pre-*-representation $\pi : A \to L(\mathscr{V})$.

Easy examples of automatically bounded elements in a *-algebra are unitaries, projections, or more generally, partial isometries.

Given a *-algebra A let

$$A_b := \{a \in A : a \text{ is automatically bounded}\} \, . \tag{1.1}$$

Definition 1.1.2 ([35], Def. 10.1.17). A *-algebra A is called a *BG^*-algebra* if every element $a \in A$ is automatically bounded, i.e. if $A_b = A$. Equivalently, A is a *BG^*-algebra* if all pre-*-representations of A are normed.

The function $\| \cdot \|_u : A \to \mathbb{R}_0^+ \cup \{\infty\}$ defined by

$$\|a\|_u := \sup_\pi \|\pi(a)\| \,, \tag{1.2}$$

where the supremum is taken over all *-representations of A, will be called the *universal norm* of A. An element $a \in A$ will be said to have a *bounded universal norm* if $\|a\|_u < \infty$, and the set of all elements $a \in A$ which have a bounded universal norm will be denoted by A_u, i.e.

$$A_u := \{a \in A : \|a\|_u < \infty\} \,. \tag{1.3}$$

When $A_u = A$ the universal norm becomes a true C^*-seminorm, being actually the largest possible C^*-seminorm in A. The Hausdorff completion of A in the universal norm is then a C^*-algebra called the *enveloping C^*-algebra* of A, which enjoys a number of universal properties (see [35, Theorem 10.1.11] and [35, Theorem 10.1.12]). For this reason, when every element $a \in A$ has a bounded universal norm, i.e. $A_u = A$, it is said that A *has an enveloping C^*-algebra*.

In general, a *-algebra does not necessarily have an enveloping C^*-algebra. Perhaps the most basic example is that of a polynomial *-algebra in a single self-adjoint variable.

We now look at the relation between automatically bounded elements and elements with a bounded universal norm. It is known that every BG^*-algebra has an enveloping C^*-algebra ([35, Proposition 10.1.19]), and the same proof yields this slightly more general result, that an automatically bounded element has a bounded universal norm:

Proposition 1.1.3. *Let A be a *-algebra. We have that $A_b \subseteq A_u$. In particular if A is a BG^*-algebra, then A has an enveloping C^*-algebra.*

Proof: Suppose $a \notin A_u$. Then there is a sequence of representations $\{\pi_i\}_{i \in \mathbb{N}}$ of A on Hilbert spaces $\{\mathscr{H}_i\}_{i \in \mathbb{N}}$, such that $\|\pi_i(a)\| \to \infty$. Consider now the inner product space $\mathscr{V}$ defined as the algebraic direct sum

$$\mathscr{V} := \bigoplus_{i \in \mathbb{N}} \mathscr{H}_i \,,$$

and the pre-*-representation $\pi := \bigoplus_{i \in \mathbb{N}} \pi_i$ of A on $L(\mathscr{V})$. It is clear by construction that $\pi(a) \notin B(\mathscr{V})$. Hence, $a \notin A_b$. $\square$

1.2 Hecke algebras

1.2.1 Left and double coset spaces

The two results in this subsection will not be used until the second part of this thesis, but it is important to establish now some notation and conventions concerning left coset spaces and double coset spaces.

Let G be a group, B, C subgroups of G and $e \in G$ the identity element. The double coset space $B\backslash G/C$ is the set

$$B\backslash G/C := \{BgC \subseteq G : g \in G\} . \tag{1.4}$$

It is easy to see that the sets of the form BgC are either equal or disjoint, or in other words, we have an equivalence relation defined in G whose equivalence classes are precisely the sets BgC.

The left coset space G/C is the set

$$G/C := \{e\}\backslash G/C = \{gC \subseteq G : g \in G\} . \tag{1.5}$$

Given an element $g \in G$ and a double coset space $B\backslash G/C$ (which can in particular be a left coset space by taking $B = \{e\}$) we will denote by $[g]$ the double coset BgC. Thus, $[g]$ denotes the whole equivalence class for which $g \in G$ is a representative.

If A is a subset of G we define the double coset space $B\backslash A/C$ as the set of double cosets in $B\backslash G/C$ which have a representative in A, i.e.

$$B\backslash A/C := \{BaC \subseteq G : a \in A\} . \tag{1.6}$$

Proposition 1.2.1. *Let A, B and C be subgroups of a group G. If $C \subseteq A$, then the following map is a bijective correspondence between the double coset spaces:*

$$\begin{aligned} B\backslash A/C &\longrightarrow (B \cap A)\backslash A/C \\ [a] &\mapsto [a] . \end{aligned} \tag{1.7}$$

Similarly, if $B \subseteq A$, then the following map is a bijective correspondence:

$$\begin{aligned} B\backslash A/C &\longrightarrow B\backslash A/(A \cap C) \\ [a] &\mapsto [a] . \end{aligned} \tag{1.8}$$

Proof: We first need to show that the map (1.7) is well defined, i.e. if $Ba_1C = Ba_2C$, for some $a_1, a_2 \in A$, then $(B \cap A)a_1C = (B \cap A)a_2C$. If $Ba_1C = Ba_2C$ then there exist $b \in B$ and $c \in C$ such that $a_1 = ba_2c$, from which it follows that $b = a_1c^{-1}a_2^{-1}$. Since A is a subgroup and $C \subseteq A$, it follows readily that $b \in B \cap A$, and therefore $a_1 \in (B \cap A)a_2C$, i.e. $(B \cap A)a_1C = (B \cap A)a_2C$.

The map defined in (1.7) is clearly surjective. It is also injective because if $(B \cap A)a_1C = (B \cap A)a_2C$, then clearly $Ba_1C = Ba_2C$.

A completely analogous argument shows that map defined in (1.8) is a bijection. □

Suppose a group G acts (on the right) on a set X and let $x \in X$. We will henceforward denote by $\mathcal{S}_x$ the stabilizer of the point x, i.e.

$$\mathcal{S}_x := \{g \in G : xg = x\}\,. \tag{1.9}$$

Given a subset $Z \subseteq X$ and a subgroup $H \subseteq G$ we denote by Z/H the set of H-orbits which have representatives in Z, i.e.

$$Z/H := \{zH : z \in Z\}\,.$$

Suppose now that $H, K \subseteq G$ are subgroups and let $x \in X$ be a point. The following result establishes a correspondence between the set of H-orbits $(xK)/H$ and the double coset space $\mathcal{S}_x\backslash K/H$:

Proposition 1.2.2. *Let G be a group which acts (on the right) on a set X. Let $x \in X$ be a point and $H, K \subseteq G$ be subgroups. We have a bijection*

$$(xK)/H \longrightarrow \mathcal{S}_x\backslash K/H\,,$$

given by $xgH \mapsto \mathcal{S}_x gH$, where $g \in K$.

Proof: Let us first prove that the map $xgH \mapsto \mathcal{S}_x gH$ is well defined, i.e. if $xg_1H = xg_2H$, then $\mathcal{S}_x g_1H = \mathcal{S}_x g_2H$. If $xg_1H = xg_2H$, then there exists $h \in h$ such that $xg_1 = xg_2h$, which implies that $x = xg_2hg_1^{-1}$, from which it follows that $g_2hg_1^{-1} \in \mathcal{S}_x$. Thus we see that

$$\mathcal{S}_x g_1 H \;=\; \mathcal{S}_x g_2 h g_1^{-1} g_1 H \;=\; \mathcal{S}_x g_2 H\,.$$

We conclude that the map is well-defined. The map is obviously surjective. It is also injective because if $\mathcal{S}_x g_1H = \mathcal{S}_x g_2H$, then there exists $r \in \mathcal{S}_x$ and $h \in H$ such that $g_1 = rg_2h$, from which it follows that $xg_1H = xrg_2hH = xg_2H$. □

1.2.2 Generalities about Hecke pairs and Hecke algebras

We will mostly follow [26] and [24] in what regards Hecke pairs and Hecke algebras and refer to these references for more details.

We start by establishing some notation which will be useful later on. Given a group G, a subgroup $\Gamma \subseteq G$ and $g \in G$, we will denote by Γ^g the subgroup

$$\Gamma^g := \Gamma \cap g\Gamma g^{-1} \,. \tag{1.10}$$

We now recall the definition of a Hecke pair:

Definition 1.2.3. Let G be a group and Γ a subgroup. The pair (G, Γ) is called a *Hecke pair* if every double coset $\Gamma g \Gamma$ is the union of finitely many right (and left) cosets. In this case, Γ is also called a *Hecke subgroup* of G.

Given a Hecke pair (G, Γ) we will denote by L and R, respectively, the left and right coset counting functions, i.e.

$$L(g) := |\Gamma g \Gamma / \Gamma| = [\Gamma : \Gamma^g] < \infty \tag{1.11}$$

$$R(g) := |\Gamma \backslash \Gamma g \Gamma| = [\Gamma : \Gamma^{g^{-1}}] < \infty \,. \tag{1.12}$$

We recall that L and R are Γ-biinvariant functions which satisfy $L(g) = R(g^{-1})$ for all $g \in G$. Moreover, the function $\Delta : G \to \mathbb{Q}^+$ given by

$$\Delta(g) := \frac{L(g)}{R(g)} \,, \tag{1.13}$$

is a group homomorphism, usually called the *modular function* of (G, Γ).

Definition 1.2.4. Given a Hecke pair (G, Γ), the *Hecke algebra* $\mathcal{H}(G, \Gamma)$ is the *-algebra of finitely supported $\mathbb{C}$-valued functions on the double coset space $\Gamma \backslash G / \Gamma$ with the product and involution defined by

$$(f_1 * f_2)(\Gamma g \Gamma) := \sum_{h\Gamma \in G/\Gamma} f_1(\Gamma h \Gamma) f_2(\Gamma h^{-1} g \Gamma) \,, \tag{1.14}$$

$$f^*(\Gamma g \Gamma) := \Delta(g^{-1}) \overline{f(\Gamma g^{-1} \Gamma)} \,. \tag{1.15}$$

Remark 1.2.5. Some authors, including Krieg [26], do not include the factor Δ in the involution. Here we adopt the convention of Kaliszewski, Landstad and Quigg [24] in doing so, as it gives rise to a more natural L^1-norm. We note, nevertheless, that there is no loss (or gain) in doing so, because these two different involutions give rise to *-isomorphic Hecke algebras. In particular, the question of existence of an enveloping C^*-algebra is not perturbed by this.

The Hecke algebra has a natural basis, as a vector space, given by the characteristic functions of double cosets. We will henceforward identify a characteristic function of a double coset $1_{\Gamma g \Gamma}$ with the double coset $\Gamma g \Gamma$ itself. It will be useful to know how to write a product $\Gamma g \Gamma * \Gamma h \Gamma$ of two double cosets in the unique linear combination of double cosets:

Lemma 1.2.6. *The expression for the product $\Gamma g \Gamma * \Gamma h \Gamma$ of two double cosets in the unique linear combination of double cosets is given by:*

$$\Gamma g \Gamma * \Gamma h \Gamma = \sum_{\Gamma s \Gamma \in \Gamma \backslash G / \Gamma} \frac{L(g)\, C_{g,h}(s)}{L(s)}\, \Gamma s \Gamma \,,$$

where $C_{g,h}(s) := \#\{w\Gamma \subseteq \Gamma h \Gamma : \Gamma g w \Gamma = \Gamma s \Gamma\}$.

Proof: Let us first check that $C_{g,h}(s)$ is well-defined. It is clear that $C_{g,h}(s)$ does not depend on the representatives h and s of the chosen double cosets, so it remains to verify that it is also independent on g. Given any other representative $\beta g \gamma$ of the double coset $\Gamma g \Gamma$, with $\beta, \gamma \in \Gamma$, it is not difficult to see that the map

$$w\Gamma \mapsto \gamma^{-1} w \Gamma \,,$$

gives a bijective correspondence between the sets $\{w\Gamma \subseteq \Gamma h \Gamma : \Gamma g w \Gamma = \Gamma s \Gamma\}$ and $\{u\Gamma \subseteq \Gamma h \Gamma : \Gamma \beta g \gamma u \Gamma = \Gamma s \Gamma\}$. Hence we have $C_{\beta g \gamma, h}(s) = C_{g,h}(s)$.

Now, to check the product formula we recall (for example from [24]) that

$$\Gamma g \Gamma * \Gamma h \Gamma = \sum_{w\Gamma \in \Gamma h \Gamma / \Gamma} \frac{L(g)}{L(gw)}\, \Gamma g w \Gamma \,, \tag{1.16}$$

where the sum runs over a set of representatives for left cosets in $\Gamma h \Gamma$. Let us fix a representative g for the double coset $\Gamma g \Gamma$ and let S be the set of double cosets $S := \{\Gamma g w \Gamma : w\Gamma \in \Gamma h \Gamma / \Gamma\}$, i.e. the set of double cosets that appear as summands in (1.16). The number of times an element $\Gamma s\Gamma \in S$ appears repeated in the sum (1.16) is precisely the number $C_{g,h}(s)$. Hence we can write

$$\Gamma g\Gamma * \Gamma h\Gamma = \sum_{\Gamma s\Gamma \in S} \frac{L(g)\, C_{g,h}(s)}{L(s)}\, \Gamma s\Gamma\,.$$

Also, if a double coset $\Gamma r\Gamma$ does not belong to S we have $C_{g,h}(r) = 0$, thus we get

$$\Gamma g\Gamma * \Gamma h\Gamma = \sum_{\Gamma s\Gamma \in \Gamma\backslash G/\Gamma} \frac{L(g)\, C_{g,h}(s)}{L(s)}\, \Gamma s\Gamma\,.$$

□

The reader can find alternative ways of describing the coefficients of this unique linear combination in [26, Lemma 4.4]. In particular, the characterization *(iii)* of the cited lemma is very similar to the one we just described.

Remark 1.2.7. A direct computation or Lemma 1.2.6 imply that the double cosets that appear in the expression for $\Gamma g\Gamma * \Gamma h\Gamma$ as a unique linear combination of double cosets are all of the form $\Gamma g\gamma h\Gamma$, for some $\gamma \in \Gamma$. Conversely, all the (necessarily finitely many) double cosets of the form $\Gamma g\gamma h\Gamma$, with $\gamma \in \Gamma$, appear in this linear combination, because $C_{g,h}(g\gamma h) \neq 0$.

Given a Hecke pair (G, Γ), the subgroup $R^\Gamma := \bigcap_{g\in G} g\Gamma g^{-1}$ is a normal subgroup of G contained in Γ. A Hecke pair (G, Γ) is called *reduced* if $R^\Gamma = \{e\}$. As it is known, the pair $(G_r, \Gamma_r) := (G/R^\Gamma, \Gamma/R^\Gamma)$ is a reduced Hecke pair and the Hecke algebras $\mathcal{H}(G, \Gamma) \cong \mathcal{H}(G_r, \Gamma_r)$ are canonically isomorphic. For this reason the pair (G_r, Γ_r) is called the *reduction* of (G, Γ), and the isomorphism of the corresponding Hecke algebras shows that it is enough to consider reduced Hecke pairs, a convention used by several authors. We will not use this convention however, since we aim at achieving general results based on properties of the original Hecke pair (G, Γ), and not its reduction.

A natural example of a Hecke pair (G, Γ) is given by a topological group G and a compact open subgroup Γ. It is known that this type of examples are, in some sense, the general case: there is a canonical construction which associates to a given reduced Hecke pair (G, Γ) a new Hecke pair $(\overline{G}, \overline{\Gamma})$ with the following properties:

1. $\overline{G}$ is a totally disconnected locally compact group;

2. $\overline{\Gamma}$ is a compact open subgroup;

3. the pair $(\overline{G}, \overline{\Gamma})$ is reduced;

4. There is a canonical embedding $\theta : G \to \overline{G}$ such that $\theta(G)$ is dense in $\overline{G}$ and $\theta(\Gamma)$ is dense in $\overline{\Gamma}$. Moreover, $\theta^{-1}(\overline{\Gamma}) = \Gamma$;

The pair $(\overline{G}, \overline{\Gamma})$ satisfies a well-known uniqueness property and is called the *Schlichting completion* of (G, Γ). For the details of this construction the reader is referred to [42] and [24] (see also [16] for a slightly different approach). We shall make a quick review of some known facts and we refer to the previous references for all the details.

Henceforward we will not write explicitly the canonical homomorphism θ, and we will instead see G as a dense subgroup of $\overline{G}$, identified with the image $\theta(G)$. The Schlichting completion $(\overline{G}, \overline{\Gamma})$ of a reduced Hecke pair (G, Γ) satisfies the following additional property:

5. there are canonical bijections $G/\Gamma \to \overline{G}/\overline{\Gamma}$ and $\Gamma\backslash G/\Gamma \to \overline{\Gamma}\backslash\overline{G}/\overline{\Gamma}$ given respectively by $g\Gamma \to g\overline{\Gamma}$ and $\Gamma g\Gamma \to \overline{\Gamma}g\overline{\Gamma}$.

If a Hecke pair (G, Γ) is not reduced, its *Schlichting completion* $(\overline{G}, \overline{\Gamma})$ is defined as the completion $(\overline{G_r}, \overline{\Gamma_r})$ of its reduction. There is then a canonical map with dense image $G \to \overline{G}$ which factors through G_r, and this map is an embedding if and only if (G, Γ) is reduced, i.e. $G \cong G_r$.

Following [24], we consider the normalized Haar measure μ on $\overline{G}$ (so that $\mu(\overline{\Gamma}) = 1$) and define the Banach *-algebra $L^1(\overline{G})$ with the usual convolution product and involution. We denote by p the characteristic function of $\overline{\Gamma}$, i.e. $p := 1_{\overline{\Gamma}}$, which is a projection in $C_c(\overline{G})$. Recalling [42] or [24], we always have canonical *-isomorphisms:

$$\mathcal{H}(G, \Gamma) \cong \mathcal{H}(G_r, \Gamma_r) \cong \mathcal{H}(\overline{G}, \overline{\Gamma}) \cong pC_c(\overline{G})p\,. \tag{1.17}$$

The modular function Δ of a reduced Hecke pair (G, Γ), defined by (1.13), is simply the modular function of the group $\overline{G}$ restricted to G.

1.2.3 L^1- and C^*-completions

As it is known, group algebras have two canonical C^*-completions, the reduced group C^*-algebra $C^*_r(G)$ and the full group C^*-algebra $C^*(G)$. For Hecke algebras the situation becomes more complicated, there being essentially four canonical C^*-completions. We will review these completions in this subsection, but first we need to recall the definitions and basic facts about regular representations of Hecke algebras and L^1-norms.

Definition 1.2.8. Let (G, Γ) be a Hecke pair. The mapping $\rho : \mathcal{H}(G, \Gamma) \to B(\ell^2(G/\Gamma))$ defined, for $f \in \mathcal{H}(G, \Gamma)$, $\xi \in \ell^2(G/\Gamma)$ and $g\Gamma \in G/\Gamma$, by

$$(\rho(f)\xi)\,(g\Gamma) := \sum_{[h] \in G/\Gamma} \Delta(h)^{\frac{1}{2}} f(\Gamma h \Gamma) \xi(gh\Gamma)\,, \tag{1.18}$$

is called the *right regular representation* of $\mathcal{H}(G, \Gamma)$.

It can be checked that ρ does define a *-representation of $\mathcal{H}(G, \Gamma)$. For the canonical vectors $\delta_{r\Gamma} \in \ell^2(G/\Gamma)$, expression (1.18) is given by:

$$\rho(f)\delta_{r\Gamma} = \sum_{[g] \in G/\Gamma} \Delta(g^{-1}r)^{\frac{1}{2}} f(\Gamma g^{-1} r \Gamma) \delta_{g\Gamma}\,, \tag{1.19}$$

and furthermore for f of the form $f = \Gamma d \Gamma$ we obtain:

$$\rho(\Gamma d \Gamma)\delta_{r\Gamma} = \sum_{t\Gamma \subseteq \Gamma d^{-1} \Gamma} \Delta(d)^{\frac{1}{2}} \delta_{rt\Gamma} \;=\; \Delta(d)^{\frac{1}{2}} \delta_{r\Gamma d^{-1}\Gamma}\,. \tag{1.20}$$

It can be easily checked, applying (1.19) to the vector δ_Γ for example, that ρ always defines a faithful *-representation.

One could in a similar fashion define a left regular representation of $\mathcal{H}(G, \Gamma)$, but in this work, however, it is the right regular representation the one that will play a central role.

There are several ways of defining a L^1-norm in a Hecke algebra. One approach is to simply take the L^1-norm from $L^1(\overline{G})$, since the isomorphism in (1.17) enables us to see the Hecke algebra as a subalgebra of $L^1(\overline{G})$. The completion of $\mathcal{H}(G, \Gamma)$ with respect to this L^1-norm is isomorphic to the corner $pL^1(\overline{G})p$. Alternatively, one may take the following definition:

Definition 1.2.9. The L^1*-norm* on $\mathcal{H}(G, \Gamma)$, denoted $\| \cdot \|_{L^1}$, is given by

$$\|f\|_{L^1} := \sum_{\Gamma g \Gamma \in \Gamma \backslash G / \Gamma} |f(\Gamma g \Gamma)|\, L(g)\,. \tag{1.21}$$

We will denote by $L^1(G, \Gamma)$ the completion of $\mathcal{H}(G, \Gamma)$ under this norm.

As observed in [42] or [24], the two L^1-norms described above are the same. In fact we have canonical *-isomorphisms

$$L^1(G,\Gamma) \cong L^1(\overline{G},\overline{\Gamma}) \cong pL^1(\overline{G})p\,. \tag{1.22}$$

There are several canonical C^*-completions of $\mathcal{H}(G,\Gamma)$. These are:

- $C_r^*(G,\Gamma)$ - Called the *reduced Hecke C^*-algebra*, it is the completion of $\mathcal{H}(G,\Gamma)$ under the C^*-norm arising from the right regular representation.
- $pC^*(\overline{G})p$ - The corner of the full group C^*-algebra $C^*(\overline{G})$.
- $C^*(L^1(G,\Gamma))$ - The enveloping C^*-algebra of $L^1(G,\Gamma)$.
- $C^*(G,\Gamma)$ - The enveloping C^*-algebra (if it exists!) of $\mathcal{H}(G,\Gamma)$. When it exists, it is usually called the *full Hecke C^*-algebra*.

The various C^*-completions of $\mathcal{H}(G,\Gamma)$ are related in the following way, through canonical surjective maps:

$$C^*(G,\Gamma) \dashrightarrow C^*(L^1(G,\Gamma)) \longrightarrow pC^*(\overline{G})p \longrightarrow C_r^*(G,\Gamma)\,.$$

As was pointed out by Hall in [18, Proposition 2.21], the full Hecke C^*-algebra $C^*(G,\Gamma)$ does not have to exist in general, with the Hecke algebra of the pair $(SL_2(\mathbb{Q}_p), SL_2(\mathbb{Z}_p))$ being one such example, where p is a prime number and $\mathbb{Q}_p$, $\mathbb{Z}_p$ denote respectively the field of p-adic numbers and the ring of p-adic integers. Nevertheless, the existence of $C^*(G,\Gamma)$ has been established for several classes of Hecke pairs (see, for example, [24] and [18]).

The question of whether some of these completions are actually the same has also been explored in the literature ([6], [24], [42]). We review here some of the main results.

The question of when one has the isomorphism $pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$ was clarified by Tzanev, in [42, Proposition 5.1], to be a matter of amenability. As pointed out in [24], there was a mistake in Tzanev's article (where it is assumed without proof that $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ is always true) which carries over to the cited Proposition 5.1. Nevertheless, Tzanev's proof holds if one just replaces $C^*(L^1(G,\Gamma))$ with $pC^*(\overline{G})p$. In order to state Tzanev's result correctly we need to recall the notion of amenability for homogeneous spaces: a pair (G,H) consisting of a locally compact group G and a closed subgroup H is said to be *amenable*, in the sense of Eymard [13], if there exists a left G-invariant mean on $L^\infty(G/H,\nu)$ where ν is a quasi-invariant measure on G/H. The correct statement of Tzanev's result is then:

Theorem 1.2.10 (Tzanev). *The following statements are equivalent:*

i) $pC^*(\overline{G})p \cong C_r^*(G, \Gamma)$.

ii) The pair (G, Γ) *is amenable.*

iii) $\overline{G}$ *is amenable.*

A known result concerning the isomorphism $C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p$ was obtained by Kaliszewski, Landstad and Quigg in [24, Theorem 5.14], where they showed that this isomorphism holds when $\overline{G}$ is a Hermitian group.

An important result of Kaliszewski, Landstad and Quigg regarding the existence of $C^*(G, \Gamma)$ and the simultaneous isomorphisms $C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p$ will be discussed in the next subsection.

1.2.4 Representation theory

As it is well-known, for any group G there is a canonical bijective correspondence (i.e. category equivalence) between unitary representations of G and nondegenerate *-representations of the group algebra $\mathbb{C}(G)$. Hall [18] asked whether something analogous was true for Hecke pairs, and the following definition is necessary in order to understand Hall's question:

Definition 1.2.11. Let G be a group and $\Gamma \subseteq G$ a subgroup. A unitary representation $\pi : G \to U(\mathscr{H})$ is said to be *generated by its* Γ*-fixed vectors* if $\overline{\pi(G)\mathscr{H}^\Gamma} = \mathscr{H}$, where $\mathscr{H}^\Gamma = \{\xi \in \mathscr{H} : \pi(\gamma)\xi = \xi\,, \text{ for all } \gamma \in \Gamma\}$.

The question which Hall posed in [18] is the following:

Question 1.2.12 (Hall's equivalence). Let (G, Γ) be a Hecke pair. Is there a category equivalence between nondegenerate *-representations of $\mathcal{H}(G, \Gamma)$ and unitary representations of G generated by the Γ-fixed vectors?

Whenever there is an affirmative answer to this question, we shall say the Hecke pair (G, Γ) satisfies *Hall's equivalence*. In the work of Hall [18] and the subsequent work of Glöckner and Willis [16], Hall's equivalence was studied and proven to hold under a certain form of positivity for some *-algebraic bimodules. A more complete approach was further developed by Kaliszewski, Landstad and Quigg in [24], where Hall's equivalence, positivity for certain *-algebraic bimodules, and C^*-completions of Hecke algebras were all shown to be related. We briefly describe here the approach and results of [24] and the reader is referred to this reference for more details.

Let $(\overline{G}, \overline{\Gamma})$ be the Schlichting completion of a Hecke pair (G, Γ). Following [24, Section 5], we have an inclusion of two imprimitivity bimodules (in Fell's sense):

$$ {}_{C_c(\overline{G})pC_c(\overline{G})}\big(C_c(\overline{G})p\big)_{\mathcal{H}(\overline{G},\overline{\Gamma})} \qquad \subseteq \qquad {}_{L^1(\overline{G})pL^1(\overline{G})}\big(L^1(\overline{G})p\big)_{L^1(\overline{G},\overline{\Gamma})}\,, $$

where the left and right inner products, $\langle\rangle_L$ and $\langle\rangle_R$, on these bimodules are given by multiplication within $L^1(\overline{G})$ by

$$ \langle f\,,\,g\rangle_L = f * g^*\,, \qquad\qquad \langle f\,,\,g\rangle_R = f^* * g\,. $$

A *-representation π of $\mathcal{H}(G,\Gamma)$ is said to be $\langle\rangle_R$-*positive* if

$$ \pi(\langle f\,,\,f\rangle_R) \geq 0\,, \qquad \text{for all } \; f \in C_c(\overline{G})p\,. \tag{1.23} $$

Similarly, a *-representation π of $L^1(G,\Gamma)$ is said to be $\langle\rangle_R$-*positive* when condition (1.23) holds for all $f \in L^1(\overline{G})p$.

In [24, Corollary 5.19] Kaliszewski, Landstad and Quigg proved that, for a reduced pair (G,Γ), there exists a category equivalence between unitary representations of G generated by the Γ-fixed vectors and the $\langle\rangle_R$-positive representations of $\mathcal{H}(G,\Gamma)$. This is in fact true for non-reduced Hecke pairs (G,Γ) as well, as follows from the following observation:

Proposition 1.2.13. *Let (G,Γ) be a Hecke pair and (G_r,Γ_r) its reduction. There exists a category equivalence between unitary representations of G generated by the Γ-fixed vectors and unitary representations of G_r generated by the Γ_r-fixed vectors.*

The correspondence is as follows: a representation $\pi : G_r \to U(\mathscr{H})$ is mapped to the representation $\pi \circ q$, where $q : G \to G_r$ is the quotient map. Its inverse map takes a representation $\rho : G \to U(\mathscr{H})$ to the representation $\widetilde{\rho}$ of G_r on the same Hilbert space, given by $\widetilde{\rho}([g]) := \rho(g)$.

Proof: First we observe that the assignment $\pi \mapsto \pi \circ q$ does indeed produce a unitary representation of G generated by the Γ-fixed vectors. This is obvious since the spaces of fixed vectors $\mathscr{H}^{\Gamma_r}$ and $\mathscr{H}^{\Gamma}$ are the same.

Secondly, for the inverse assignment, we need to check that $\widetilde{\rho}$ is well-defined, which amounts to show that $\rho(g) = \rho(gh)$ for any $g \in G$ and $h \in R^\Gamma$. For any $s \in G$ and $\xi \in \mathscr{H}^\Gamma$ we have

$$\begin{aligned} \rho(gh)\rho(s)\xi &= \rho(g)\rho(s)\rho(s^{-1}hs)\xi \\ &= \rho(g)\rho(s)\xi \,, \end{aligned}$$

because $s^{-1}hs \in R^\Gamma \subseteq \Gamma$. Hence, $\rho(gh) = \rho(g)$ on the space $\overline{\pi(G)\mathscr{H}^\Gamma}$. Since ρ is assumed to be generated by the Γ-fixed vectors, it follows that $\rho(gh) = \rho(g)$.

It is also easy to see that $\widetilde{\rho}$ is generated by the Γ_r-fixed vectors and it is clear from the definitions that these assignments are inverse of one another.

This correspondence does not change the Hilbert spaces of the representations, so that the intertwiners of representations are preserved in a canonical way. It can then be easily seen that this defines a category equivalence. $\square$

In the light of Kaliszewski, Landstad and Quigg's result, for a Hecke pair (G, Γ) for which all *-representations of $\mathcal{H}(G, \Gamma)$ are $\langle\rangle_R$-positive, there exists a category equivalence between unitary representations of G generated by the Γ-fixed vectors and nondegenerate *-representations of $\mathcal{H}(G, \Gamma)$. In other words, Hall's equivalence holds when all *-representations of $\mathcal{H}(G, \Gamma)$ are $\langle\rangle_R$-positive. Furthermore, Kaliszewski, Landstad and Quigg proved also the following relation between $\langle\rangle_R$-positivity and C^*-completions of Hecke algebras:

Theorem 1.2.14 ([24] Corollary 5.11). *Let (G, Γ) be a Hecke pair.*

1. *Every *-representation of $\mathcal{H}(G, \Gamma)$ is $\langle\rangle_R$-positive if and only if $C^*(G, \Gamma)$ exists and $C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p$.*

2. *Similarly, every *-representation of $L^1(G, \Gamma)$ is $\langle\rangle_R$-positive if and only if $C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p$.*

1.3 Directed graphs

Recall that a *simple directed graph* $\mathcal{G} := (B, E)$ consists of a set B, whose elements are called *vertices*, and a subset $E \subseteq B^2$, whose elements are called (directed) *edges*. A directed edge is thus a pair of vertices (a, b), which we see as directed from a to b. Since we are only interested in directed graphs that are simple, i.e. such that there is at most one edge directed from one vertex to another, we will henceforward drop the word *simple* and simply write *directed graph*.

Let us now set some notation. Let $\mathcal{G} := (B, E)$ be a directed graph. If the ordered pair (a, b) belongs to E we say that b is a *successor* of a.

Definition 1.3.1. Let $\mathcal{G} := (B, E)$ be a directed graph. A set of vertices $Y \subseteq B$ is said to be *co-hereditary* if it contains the successors of all of its elements, i.e. if $a \in Y$ and $b \in B$ is a successor of a, then $b \in Y$.

It is easy to see that an arbitrary intersection of co-hereditary sets is still a co-hereditary set. Hence, we can talk about the co-hereditary set generated by a subset $X \subseteq B$ of vertices:

Definition 1.3.2. Let $\mathcal{G} := (B, E)$ be a directed graph and $X \subseteq B$ a set of vertices. The *co-hereditary set generated by* X is the smallest co-hereditary set that contains X.

Given a directed graph $\mathcal{G} := (B, E)$ and a set of vertices $X \subseteq B$, we will denote by $S(X)$ the set of all the successors of all elements of X, i.e.

$$S(X) := \{a \in B : a \text{ is a successor of } x, \text{ for some } x \in X\}\,. \tag{1.24}$$

Similarly, we define the *n-th successor set of* X inductively as follows:

$$S^0(X) := X\,, \qquad\qquad S^n(X) := S(S^{n-1}(X))\,, \text{ for } n \geq 1\,. \tag{1.25}$$

We will often consider X to be a singleton set $X = \{b\}$, and in this case we will use the notation $S(b)$ instead of $S(\{b\})$. The following result follows easily from the definitions:

Lemma 1.3.3. *Let $\mathcal{G} := (B, E)$ be a directed graph and $X \subseteq B$ a set of vertices. The co-hereditary set generated by X is the set $\bigcup_{n \in \mathbb{N}_0} S^n(X)$.*

Remark 1.3.4. The sets of vertices we are going to consider in our applications will be sets with specific additional structure (for instance, the set of vertices will typically be a basis of a vector space), and we are interested in proving results of the type: all elements of the co-hereditary set generated by X have a certain property P. To do so, we use a certain form of "induction". Namely, if we prove that all elements of X have the property P, and if we prove that the property P is preserved upon taking successors, then by Lemma 1.3.3 and the usual induction on $\mathbb{N}$, all elements of the co-hereditary set generated by X will also satisfy P.

1.4 Groups of subexponential growth

Definition 1.4.1. Let G be a locally compact group with a Haar measure μ. For a compact neighbourhood V of e, the sequence $\{\mu(V^n)^{\frac{1}{n}}\}_{n\in\mathbb{N}}$ is called the *growth function* of V, and the limit superior

$$\limsup_{n\to\infty} \mu(V^n)^{\frac{1}{n}} \tag{1.26}$$

will be called the *growth rate* of V.

It was shown by Guivarc'h [17, Théorème I.1] that if G is compactly generated and V is a compact neighbourhood of e that generates G, then the limsup in (1.26) is in fact a true limit and it is always finite and greater or equal to 1. This holds in fact for any locally compact group:

Proposition 1.4.2. *Let G be a locally compact group and V a compact neighbourhood of e. The limit*

$$\lim_{n\to\infty} \mu(V^n)^{\frac{1}{n}}$$

always exists and is always finite and greater or equal to 1.

Proof: Let $A := V \cap V^{-1}$, which is clearly a symmetric compact neighbourhood of e. By [17, Lemme I.1] and the fact that $A \subseteq V$ we have, for any $n, m \in \mathbb{N}$, that

$$\mu(A)\mu(V^mV^n) \leq \mu(V^mA)\mu(A^{-1}V^n) \leq \mu(V^{m+1})\mu(V^{n+1})\,. \tag{1.27}$$

We notice that even though Guivarc'h is working under the assumption that G is compactly generated, the proof of [17, Lemme I.1] is completely general and holds for arbitrary locally compact groups. Using the cited lemma again and the decomposition $V^{k+1} = V^{k-1}V^2$ we see that we also have that

$$\mu(V)\mu(V^{k+1}) \leq \mu(V^{k-1}V)\mu(V^{-1}V^2) = \mu(V^k)\mu(V^{-1}V^2)\,. \tag{1.28}$$

Applying inequalities (1.27) and (1.28) we then have that

$$\begin{aligned}\mu(V^{m+n}) &\leq \frac{1}{\mu(A)}\mu(V^{m+1})\mu(V^{n+1}) \\ &\leq \frac{\mu\big(V^{-1}V^2\big)^2}{\mu(A)\mu(V)^2}\,\mu(V^m)\mu(V^n)\,.\end{aligned}$$

Hence the sequence $\{\log\big(\mu(V^n)^{\frac{1}{n}}\big)\}_{n\in\mathbb{N}}$ is a sequence that satisfies the conditions of [17, Lemme I.2] and therefore we conclude that the limit $\lim_{n\to\infty}\mu(V^n)^{\frac{1}{n}}$ exists and is finite. This limit is clearly greater or equal to 1 since $\mu(V)\leq\mu(V^n)$ for all $n\in\mathbb{N}$. □

Definition 1.4.3. A locally compact group G is said to be of

- *subexponential growth* if $\lim_{n\to\infty}\mu(V^n)^{\frac{1}{n}}=1$ for all compact neighbourhoods V of e.
- *exponential growth* if $\lim_{n\to\infty}\mu(V^n)^{\frac{1}{n}}>1$ for at least one compact neighbourhood V of e.

The class of groups with subexponential growth is closed under taking closed subgroups [17, Théorème I.2] and quotients [17, Théorème I.3]. We observe that even though in [17] the author is only working with compactly generated groups, the proofs of these results are completely general and hold for any locally compact group.

It is known that if G has subexponential growth as a discrete group, then it has subexponential growth with respect to any other locally compact topology [21, Theorem 3.1]. The following is a slight generalization of this result, and the proof is done along similar lines:

Proposition 1.4.4. *Let H be a dense subgroup of a locally compact group $\overline{H}$. If H has subexponential growth as a discrete group, then $\overline{H}$ has subexponential growth in its locally compact topology.*

Proof: Let $A\subseteq\overline{H}$ be a compact neighbourhood of e. First we claim that $HA=\overline{H}$. Since A is a neighbourhood of $\{e\}$, there is an open set $U\subseteq A$ such that $e\in U$. To show that $HA=\overline{H}$, let $g\in\overline{H}$. Since H is dense in $\overline{H}$ and $g(U\cap U^{-1})$ is open, it follows that there exists $h\in H\cap g(U\cap U^{-1})$. Thus, there exists $s \in U \cap U^{-1}$ such that $h = gs$, or equivalently, $g = hs^{-1}$. Since $s^{-1} \in U \cap U^{-1}$ we then have $g \in hU$, and thus $g \in hA$. Hence $\overline{H} = HA$.

From the previous observation it follows that $\{hA\}_{h\in H}$ is an open covering of the compact set AA, and since A has non-empty interior there must exist a finite set $F \subset H$ such that $AA \subseteq FA$. Hence, we have $A^n \subseteq F^{n-1}A$, for all $n \geq 2$. Without loss of generality we can assume that F contains the identity element. Now using the fact that H has subexponential growth we obtain

$$\lim_{n\to\infty} \mu\big(A^n\big)^{\frac{1}{n}} \;\; \leq \;\; \lim_{n\to\infty} \mu(F^{n-1}A)^{\frac{1}{n}} \;\; \leq \;\; \lim_{n\to\infty} |F^{n-1}|^{\frac{1}{n}} \mu\big(A\big)^{\frac{1}{n}} \;\; = \;\; 1\,.$$

$\square$

Corollary 1.4.5. *Let (G, Γ) be a discrete Hecke pair. If G (or G_r) has subexponential growth, then so does $\overline{G}$.*

Proof: If G has subexponential growth than so does any of its quotients, so in particular G_r also has subexponential growth. So to prove the statement of this corollary we only need to prove that if G_r has subexponential growth then so does $\overline{G}$, and that follows directly from Proposition 1.4.4. $\square$

Groups with subexponential growth are always unimodular [17, Lemme I.3] and amenable (see [35, Section 12.6.18]).

The class of groups with subexponential growth includes all locally nilpotent groups and all FC^{-}-groups [35, Theorem 12.5.17]. In particular, all abelian and all compact groups have subexponential growth.

Chapter 2

On enveloping C^*-algebras of Hecke algebras

In this chapter we address two problems: the question of existence of the full Hecke C^*-algebra $C^*(G,\Gamma)$ and the question of when does $C^*(G,\Gamma)$, provided it exists, coincide with $C^*(L^1(G,\Gamma))$. These problems will be addressed simultaneously through the same method.

In Section 2.1 we associate a directed graph to given *-algebra with a specified basis and derive a sufficient condition, based on a property of the graph, for the *-algebra to be a BG^*-algebra and therefore have an enveloping C^*-algebra. This result is sharpened in Section 2.2 for Hecke algebras, where we show that under a slightly stronger assumption on the graph we can not only show the existence of $C^*(G,\Gamma)$ but also that the isomorphism $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$ holds.

We develop some tools, based on iterated commutators on the group G, that allows us to show that the graph properties under consideration are satisfied for several classes of Hecke pairs.

2.1 Graph associated with a *-algebra

Let A be a *-algebra. Suppose that we are given a finite set of elements $\{b_1,\dots,b_n\} \subseteq A$ satisfying a set of relations of the form

$$\begin{cases} b_1^*b_1 &= \lambda_{11}b_1+\cdots+\lambda_{1n}b_n \\ & \vdots \\ b_n^*b_n &= \lambda_{n1}b_1+\cdots+\lambda_{nn}b_n\,, \end{cases} \tag{2.1}$$

where $\lambda_{ij} \in \mathbb{C}$ for each $i, j \in \{1, \ldots, n\}$. We claim that the elements $b_1, \ldots, b_n$ are automatically bounded, and this fact will pave the way for our study of existence of enveloping C^*-algebras:

Theorem 2.1.1. *Let A be a *-algebra and $\{b_1, \ldots, b_n\} \subseteq A$ a finite set of elements satisfying relations as in (2.1). Then the elements $b_1, \ldots, b_n$ are automatically bounded. In particular they have a bounded universal norm.*

In order to prove Theorem 2.1.1 we will need the following lemma:

Lemma 2.1.2. *Let $n \in \mathbb{N}$ and $k_{ij} \in \mathbb{R}_0^+$ for every $1 \leq i, j \leq n$. The set*

$$B := \{(x_1, \ldots, x_n) \in (\mathbb{R}_0^+)^n : x_i^2 \leq k_{i1}x_1 + \cdots + k_{in}x_n \qquad \forall 1 \leq i \leq n\}$$

is bounded in $\mathbb{R}^n$.

Proof: Let us denote by β the real number

$$\beta := \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{n} k_{ij}}\,,$$

and let $\widetilde{B}$ be the set defined by

$$\widetilde{B} := \left\{(x_1, \ldots, x_n) \in (\mathbb{R}_0^+)^n : x_1 + \cdots + x_n \leq \beta\, \sqrt{x_1 + \cdots + x_n}\,\right\},$$

We claim that $B \subseteq \widetilde{B}$. To see this, let $(x_1, \ldots, x_n) \in B$. We have

$$\begin{aligned}
x_1 + \cdots + x_n &\leq \sum_{i=1}^{n} \sqrt{k_{i1}x_1 + \cdots + k_{in}x_n} \\
&\leq \sum_{i=1}^{n} \sqrt{\Big(\sum_{j=1}^{n} k_{ij}\Big)x_1 + \cdots + \Big(\sum_{j=1}^{n} k_{ij}\Big)x_n} \\
&= \sum_{i=1}^{n} \sqrt{\Big(\sum_{j=1}^{n} k_{ij}\Big)\Big(x_1 + \cdots + x_n\Big)} \\
&= \beta\, \sqrt{x_1 + \cdots + x_n}\,,
\end{aligned}$$

and therefore $(x_1, \ldots, x_n) \in \widetilde{B}$.

Hence, it is enough to prove that the set $\widetilde{B}$ is bounded. As it is well known, linear functions in $\mathbb{R}$ grow faster than square roots, thus it is clear that the set

$$Y := \{x \in \mathbb{R}_0^+ : x \leq \beta\sqrt{x}\}$$

is bounded in $\mathbb{R}$. Let $S : (\mathbb{R}_0^+)^n \to \mathbb{R}$ be the function $S(x_1, \ldots, x_n) := \sum_{i=1}^n x_i$. We have that $\widetilde{B} \subseteq S^{-1}(Y)$. Since S is only defined for elements in $(\mathbb{R}_0^+)^n$, the pre-image by S of a bounded set in $\mathbb{R}$ is also a bounded set in $(\mathbb{R}_0^+)^n$. We conclude that $\widetilde{B}$, and therefore B, is bounded. $\square$

Proof of Theorem 2.1.1: Let $\{b_1, \ldots, b_n\} \subseteq A$ be a finite set in A satisfying relations as in (2.1) and $B \subseteq (\mathbb{R}_0^+)^n$ the set defined by

$$B := \{(x_1, \ldots, x_n) \in (\mathbb{R}_0^+)^n : x_i^2 \leq |\lambda_{i1}|x_1 + \cdots + |\lambda_{in}|x_n \qquad \forall 1 \leq i \leq n\}\,.$$

Let $\pi : A \to L(\mathscr{V})$ be a pre-*-representation and $\xi \in \mathscr{V}$ a vector such that $\|\xi\| = 1$. We have that

$$\begin{aligned} \|\pi(b_i)\xi\|^2 &= \langle \pi(b_i^* b_i)\xi, \xi\rangle \;\leq\; \|\pi(b_i^* b_i)\xi\|\|\xi\| \\ &= \|\pi(b_i^* b_i)\xi\| \;=\; \|\sum_{j=1}^n \lambda_{ij}\, \pi(b_j)\xi\| \\ &\leq \sum_{j=1}^n |\lambda_{ij}|\, \|\pi(b_j)\xi\|\,. \end{aligned}$$

Hence it follows that $\big(\|\pi(b_1)\xi\|, \ldots, \|\pi(b_n)\xi\|\big) \in B$. Since the definition of the set B is independent of π and ξ, and since by Lemma 2.1.2 we know that B is bounded in $\mathbb{R}^n$, it follows that

$$\sup_{\|\xi\|=1} \|\pi(b_i)\xi\| < \infty\,,$$

i.e. $\pi(b_i) \in B(\mathscr{V})$, for all $1 \leq i \leq n$, and all pre-*-representations π. Thus, the elements $b_1, \ldots, b_n$ are all automatically bounded and therefore have bounded universal norms by Proposition 1.1.3 $\square$

In practice though, Theorem 2.1.1 can be difficult to apply, as in general one is not given a set of elements $\{b_1, \ldots, b_n\}$ satisfying the prescribed relations, especially if the structure of the *-algebra A is not well understood. For this reason we will describe a more algorithmic approach to Theorem 2.1.1 where the set $\{b_1, \ldots, b_n\}$ is not given from the start, but it is instead constructed step-by-step starting from one element b_1. This method will be explained through the language of graphs and will be especially useful when applied to Hecke algebras, where knowledge from the Hecke pair can many times be used to show that sets of elements $\{b_1, \dots, b_n\}$ satisfying (2.1) abound.

Let A be a *-algebra and B a basis of A as a vector space. Given a basis element $b_0 \in B$ we will denote by Φ_{b_0} the unique linear functional $\Phi_{b_0} : A \to \mathbb{C}$ such that

$$\Phi_{b_0}(b) := \begin{cases} 1, & \text{if } b = b_0 \\ 0, & \text{if } b \neq b_0 \end{cases} \tag{2.2}$$

for every $b \in B$.

Definition 2.1.3. Given a *-algebra A with a specified basis B, we define its *associated graph* as the directed graph $\mathcal{G} := (B, E)$, whose set of vertices is the set B and whose set of edges is the set

$$E := \{(a, b) \in B^2 : \ \Phi_b\,(a^* a) \neq 0\}\,. \tag{2.3}$$

Thus, given a vertex $a \in B$, its successors are precisely those basis elements that have non-zero coefficients in the unique expression of $a^* a$ as a linear combination of elements of B, i.e. if

$$a^* a = k_1 b_1 + \cdots + k_n b_n\,,$$

where each $k_i \in \mathbb{C}$ is non-zero and the basis elements b_i are all different, then the successors of a are precisely $b_1, \dots, b_n$.

Proposition 2.1.4. *Let A be a *-algebra with basis B and $\mathcal{G}$ its associated graph. If $X \subseteq B$ is a finite co-hereditary set in $\mathcal{G}$, then all elements of X are automatically bounded. In particular, all elements of X have a bounded universal norm.*

Proof: Let $X = \{b_1, \dots, b_n\} \subseteq B$. Since X contains the successors of all its elements, we must necessarily have

$$b_i^* b_i = \lambda_{i1} b_1 + \cdots + \lambda_{in} b_n\,, \qquad 1 \leq i \leq n\,,$$

for some elements $\lambda_{ij} \in \mathbb{C}$ (possibly being zero). It then follows from Theorem 2.1.1 that all elements $b_1, \ldots, b_n$ are automatically bounded and in particular have a bounded universal norm. $\square$

Corollary 2.1.5. *Let A be a *-algebra, B a basis for A and $\mathcal{G}$ its associated graph. If A is generated as a *-algebra by the elements of the finite co-hereditary sets of $\mathcal{G}$, then A is a BG^*-algebra. In particular A has an enveloping C^*-algebra.*

Proof: Let B_0 be the set of elements of the finite co-hereditary sets of $\mathcal{G}$. By Proposition 2.1.4, all elements in B_0 are contained in A_b. Since the elements of B_0 generate the *-algebra A, we conclude that $A = A_b$, i.e. A is a BG^*-algebra. $\square$

We can interpret the above corollary in the following (equivalent) way: suppose we have a *-algebra A with a basis B. Suppose additionally that we have a particular set $B_0 \subset B$ which generates A. If all the elements of B_0 generate finite co-hereditary sets of the associated graph, then A has an enveloping C^*-algebra. Let us now give a couple of immediate examples:

Example 2.1.6. Let A be a finite-dimensional *-algebra. If we take any basis B, the associated graph necessarily has finitely many vertices (and edges). Thus, the co-hereditary set generated by any $b \in B$ is finite.

Example 2.1.7. Let G be a discrete group, $\mathbb{C}(G)$ its group algebra with basis $\{\delta_g \in \mathbb{C}(G) : g \in G\}$. Since in the group algebra we have $\delta_g^* * \delta_g = \delta_e$, the only successor of δ_g in the associated graph is δ_e. Since δ_e is the only successor of itself, the co-hereditary set generated by δ_g has only two elements, δ_g and δ_e.

Some non-trivial examples, arising from Hecke algebras, will be computed later in Section 2.4.

2.2 A sufficient condition implying the isomorphism $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$

In Corollary 2.1.5 of the previous section we obtained a sufficient condition for a *-algebra to have an enveloping C^*-algebra, namely when it is generated by elements of the finite co-hereditary sets (with respect to a given basis). In this section we will improve this result in the case of a Hecke algebra $\mathcal{H}(G,\Gamma)$: under a suitable assumption we will not only assure an enveloping C^*-algebra $C^*(G,\Gamma)$ exists, but we will also be able to identify it with $C^*(L^1(G,\Gamma))$.

Throughout this section and henceforward (G,Γ) will denote a Hecke pair. We will always consider the canonical basis in the Hecke algebra $\mathcal{H}(G,\Gamma)$, consisting of double cosets $\{\Gamma g \Gamma : g \in G\}$. This section is devoted to the proof of the following result:

Theorem 2.2.1. *Let (G,Γ) be a Hecke pair. If all double cosets generate finite co-hereditary sets, then the enveloping C^*-algebra of $\mathcal{H}(G,\Gamma)$ exists and coincides with $C^*(L^1(G,\Gamma))$.*

In order to give a proof of Theorem 2.2.1 we will make use of several lemmas.

Lemma 2.2.2. *Let (G,Γ) be a Hecke pair and $\Gamma g\Gamma, \Gamma h\Gamma$ be two double cosets. We have that the L^1-norm satisfies the equality*

$$\|\Gamma g\Gamma * \Gamma h\Gamma\|_{L^1} = \|\Gamma g\Gamma\|_{L^1}\|\Gamma h\Gamma\|_{L^1} .$$

In particular the following equality is also satisfied

$$\|(\Gamma g\Gamma)^* * \Gamma g\Gamma\|_{L^1} = \|\Gamma g\Gamma\|_{L^1}^2 .$$

Proof: Recall from Lemma 1.2.6 that the unique expression of $\Gamma g\Gamma * \Gamma h\Gamma$ as a linear combination of double cosets is given by

$$\Gamma g\Gamma * \Gamma h\Gamma = \sum_{\Gamma s\Gamma \in \Gamma\backslash G/\Gamma} \frac{L(g)\, C_{g,h}(s)}{L(s)}\, \Gamma s\Gamma\,, \tag{2.4}$$

where $C_{g,h}(s) := \#\{w\Gamma \subseteq \Gamma h\Gamma : \Gamma g w\Gamma = \Gamma s\Gamma\}$.

Let $\Gamma s_1\Gamma, \ldots, \Gamma s_n\Gamma$ be the only elements in expression (2.4) with a non-zero coefficient. We then see that

$$\begin{aligned}\|\Gamma g\Gamma * \Gamma h\Gamma\|_{L^1} &= \sum_{i=1}^{n} \frac{L(g)\, C_{g,h}(s_i)}{L(s_i)} \|\Gamma s_i\Gamma\|_{L^1} \\ &= \sum_{i=1}^{n} \frac{L(g)\, C_{g,h}(s_i)}{L(s_i)} L(s_i) \\ &= L(g) \sum_{i=1}^{n} C_{g,h}(s_i)\,.\end{aligned}$$

Now the sets $\mathfrak{C}_{g,h}(s_i) := \{w\Gamma \subseteq \Gamma h\Gamma : \Gamma g w\Gamma = \Gamma s_i\Gamma\}$ are all mutually disjoint, for $1 \leq i \leq n$, and their union is $\{w\Gamma \subseteq \Gamma h\Gamma\}$. Therefore, since $C_{g,h}(s_i) = \#\mathfrak{C}_{g,h}(s_i)$, we get

$$\begin{aligned}\|\Gamma g\Gamma * \Gamma h\Gamma\|_{L^1} &= L(g)L(h) \\ &= \|\Gamma g\Gamma\|_{L^1}\|\Gamma h\Gamma\|_{L^1}\,.\end{aligned}$$

The second claim in this lemma follows directly from the first because

$$\|(\Gamma g\Gamma)^* * \Gamma g\Gamma\|_{L^1} = \Delta(g)\|\Gamma g^{-1}\Gamma\|_{L^1}\|\Gamma g\Gamma\|_{L^1} = \|\Gamma g\Gamma\|_{L^1}^2\,.$$

$\square$

Lemma 2.2.3. *Let $n \in \mathbb{N}$ and $A = [a_{ij}]$ be an $n \times n$ matrix whose entries satisfy: $a_{ii} \in \mathbb{R}^+$ and $a_{ij} \in \mathbb{R}_0^-$ for all $i \neq j$. If there are vectors $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathbf{z} = (z_1, \ldots, z_n)$ both in $(\mathbb{R}^+)^n$ satisfying the system*

$$A\mathbf{z} = \mathbf{d}\,, \tag{2.5}$$

then A is non-singular.

Proof: Let $\mathbf{z} \in (\mathbb{R}^+)^n$ be a solution to the above system. Suppose that $\operatorname{Ker} A \neq \{0\}$. Then, the set of solutions to the system (2.5) contains a line L. Consider now the set S of all the (finitely many) points which are the intersections of L with the canonical hyperplanes of the form $x_i = 0$, and take a point $\mathbf{y} \in S$ (not necessarily unique) which is closest to $\mathbf{z}$. The point $\mathbf{y}$ is the intersection of L with one of the hyperplanes $x_i = 0$, say $x_{i_0} = 0$ with $1 \leq i_0 \leq n$. Since $\mathbf{y} = (y_1, \ldots, y_n)$ is in L, it is also a solution of the system (2.5) and therefore must satisfy

$$\sum_{\substack{k=1 \\ k \neq i_0}}^{n} a_{i_0 k} y_k = d_{i_0}\,,$$

implying that there exists at least one number y_k which is negative. But on the other hand, the open segment between $\mathbf{z}$ and $\mathbf{y}$ lies inside $(\mathbb{R}^+)^n$ because $\mathbf{z} \in (\mathbb{R}^+)^n$ and this segment does not intersect any hyperplane $x_i = 0$ (by choice of the point $\mathbf{y}$). Thus the entries of $\mathbf{y} = (y_1, \dots, y_n)$ are all non-negative, which is a contradiction. Therefore $\operatorname{Ker} A = \{0\}$. $\square$

In preparation for the next lemma we set some notation. Given two vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$, we will write $\mathbf{a} \leq \mathbf{b}$ whenever $a_i \leq b_i$ for every $1 \leq i \leq n$. We will denote the zero vector by $\mathbf{0} = (0, \dots, 0)$. Also, given a set of vectors $S \subseteq \mathbb{R}^n$, we will denote by $\mathcal{C}(S)$ the *cone generated by* S, i.e. the set of all linear combinations with coefficients in $\mathbb{R}_0^+$ of the elements of S.

Lemma 2.2.4. *Let $n \in \mathbb{N}$ and $A = [a_{ij}]$ be an $n \times n$ matrix whose entries satisfy: $a_{ii} \in \mathbb{R}^+$ and $a_{ij} \in \mathbb{R}_0^-$ for all $i \neq j$. Assume that there are vectors $\mathbf{d} = (d_1, \dots, d_n) > \mathbf{0}$ and $\mathbf{z} = (z_1, \dots, z_n) > \mathbf{0}$ satisfying the system $A\mathbf{z} = \mathbf{d}$. Then, if*

$$A\mathbf{y} \geq \mathbf{0},$$

for some $\mathbf{y} \in \mathbb{R}^n$, we must have $\mathbf{y} \geq \mathbf{0}$.

Proof: As we are in the conditions of Lemma 2.2.3, the matrix A is nonsingular. First we claim that $\{\mathbf{y} : A\mathbf{y} \geq \mathbf{0}\} = \mathcal{C}(A^{-1}\mathbf{e}_1, \dots, A^{-1}\mathbf{e}_n)$, where $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{R}^n$ are the canonical unit vectors. The inclusion $\supseteq$ is obvious, while the inclusion $\subseteq$ follows from the fact that if $A\mathbf{y} \geq \mathbf{0}$ then we can write $A\mathbf{y}$ as a positive linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$. Thus, to prove this lemma it suffices to prove that $A^{-1}\mathbf{e}_k \geq \mathbf{0}$ for every $1 \leq k \leq n$, and we will show this by induction on n. The case $n = 1$ is obvious since $a_{11} \in \mathbb{R}^+$. Let us now assume that the result holds for $n-1$, and prove it for n. Let B_k be the matrix obtained from A by deleting the k-th row and column. Since $A\mathbf{z} = \mathbf{d}$, it follows readily that

$$B_k \begin{pmatrix} z_1 \\ \vdots \\ z_{k-1} \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} d_1 - a_{1k} z_k \\ \vdots \\ d_{k-1} - a_{k-1\,k} z_k \\ d_{k+1} - a_{k+1\,k} z_k \\ \vdots \\ d_n - a_{n\,k} z_k \end{pmatrix} \tag{2.6}$$

Since the right hand side of (2.6) is a vector in $(\mathbb{R}^+)^{n-1}$, and moreover the entries of the matrix B_k satisfy the conditions in the statement of the lemma, we can use the induction hypothesis on the matrix B_k. Let $\mathbf{v} := (v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n) \in \mathbb{R}^{n-1}$ be a solution to the equation

$$B_k\,\mathbf{v} = \begin{pmatrix} d_1 \\ \vdots \\ d_{k-1} \\ d_{k+1} \\ \vdots \\ d_n \end{pmatrix},$$

which exists by Lemma 2.2.3 (the reason for the chosen indexing of the entries of $\mathbf{v}$ will become clear in the remaining part of the proof). The induction hypothesis tells us that $\mathbf{v} \geq \mathbf{0}$. We also have that

$$B_k \begin{pmatrix} z_1 \\ \vdots \\ z_{k-1} \\ z_{k+1} \\ \cdots \\ z_n \end{pmatrix} - B_k\,\mathbf{v} = \begin{pmatrix} d_1 - a_{1k}z_k \\ \vdots \\ d_{k-1} - a_{k-1\,k}z_k \\ d_{k+1} - a_{k+1\,k}z_k \\ \vdots \\ d_n - a_{n\,k}z_k \end{pmatrix} - \begin{pmatrix} d_1 \\ \vdots \\ d_{k-1} \\ d_{k+1} \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} -a_{1k}z_k \\ \vdots \\ -a_{k-1\,k}z_k \\ -a_{k+1\,k}z_k \\ \vdots \\ -a_{n\,k}z_k \end{pmatrix} \geq \mathbf{0}\,.$$

By the induction hypothesis again, we have $z_i - v_i \geq 0$, for $i \neq k$.

Consider now the vector $\widetilde{\mathbf{v}} \in \mathbb{R}^n$ given by $\widetilde{\mathbf{v}} := (v_1, \ldots, v_{k-1}, 0, v_{k+1}, \ldots, v_n)$. We have that

$$A\,(\mathbf{z} - \widetilde{\mathbf{v}}) = \begin{pmatrix} d_1 \\ \vdots \\ d_{k-1} \\ d_k \\ d_{k+1} \\ \vdots \\ d_n \end{pmatrix} - \begin{pmatrix} d_1 \\ \vdots \\ d_{k-1} \\ \sum_{i\neq k}^n a_{ki}v_i \\ d_{k+1} \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_k - \sum_{i\neq k}^n a_{ki}v_i \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

or in other words,

$$\mathbf{z} - \widetilde{\mathbf{v}} = A^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_k - \sum_{i \neq k}^n a_{ki} v_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (d_k - \sum_{i \neq k}^n a_{ki} v_i)\; A^{-1} \mathbf{e}_k \,.$$

We now notice that $d_k - \sum_{i \neq k}^n a_{ki} v_i > 0$, because all the $a_{ki} \in \mathbb{R}_0^-$ for $k \neq i$, $v_i \geq 0$ as we saw before, and $d_k > 0$. We have already proven that $z_i - v_i \geq 0$, for $i \neq k$, from which it readily follows that $\mathbf{z} - \widetilde{\mathbf{v}} \geq \mathbf{0}$. We can now conclude that $A^{-1}\mathbf{e}_k = \frac{1}{d_k - \sum_{i \neq k}^n a_{ki} v_i} (\mathbf{z} - \widetilde{\mathbf{v}}) \geq \mathbf{0}$. $\square$

Proof of Theorem 2.2.1: We already know that if all double cosets generate finite co-hereditary sets, then $\mathcal{H}(G, \Gamma)$ has an enveloping C^*-algebra. Thus, it remains to see that this enveloping C^*-algebra is the enveloping C^*-algebra of $L^1(G, \Gamma)$, and for this we only need to show that

$$\|a\|_u \leq \|a\|_{L^1}, \tag{2.7}$$

for any $a \in \mathcal{H}(G, \Gamma)$. Actually we only need to prove (2.7) when a is a double coset $a = \Gamma s \Gamma$, since the result for a general $a \in \mathcal{H}(G, \Gamma)$ follows from the following argument: if we write a in the unique linear combination of double cosets, $a = \sum_{i=1}^n \lambda_i \Gamma s_i \Gamma$, we then have

$$\begin{aligned} \|a\|_u &= \|\sum_{i=1}^n \lambda_i \Gamma s_i \Gamma\|_u \;\leq\; \sum_{i=1}^n |\lambda_i| \|\Gamma s_i \Gamma\|_u \\ &\leq \sum_{i=1}^n |\lambda_i| \|\Gamma s_i \Gamma\|_{L^1} \;=\; \|\sum_{i=1}^n \lambda_i \Gamma s_i \Gamma\|_{L^1} \\ &= \|a\|_{L^1} \,. \end{aligned}$$

Let therefore $\Gamma s \Gamma$ be a double coset and $\{\Gamma s_1 \Gamma, \ldots, \Gamma s_n \Gamma\}$ the finite co-hereditary set it generates. By Lemma 1.2.6 we have

$$(\Gamma s_i \Gamma)^* * \Gamma s_i \Gamma = \sum_{j=1}^n \lambda_{ij}\, \Gamma s_j \Gamma \,, \tag{2.8}$$

where the coefficients λ_{ij} are given by

$$\lambda_{ij} \;:=\; \Delta(s_i) \frac{L(s_i^{-1}) C_{s_i^{-1}, s_i}(s_j)}{L(s_j)} \;=\; \frac{L(s_i) C_{s_i^{-1}, s_i}(s_j)}{L(s_j)} \,.$$

Let B be the set

$$B := \{(x_1, \ldots, x_n) \in (\mathbb{R}_0^+)^n : x_i^2 \leq \lambda_{i1}x_1 + \cdots + \lambda_{in}x_n\,, \quad \forall\, 1 \leq i \leq n\}\,.$$

Let us also denote by C the subset of B determined by

$$C := \{(x_1, \ldots, x_n) \in (\mathbb{R}_0^+)^n : x_i^2 = \lambda_{i1}x_1 + \cdots + \lambda_{in}x_n\,, \quad \forall\, 1 \leq i \leq n\}\,.$$

It follows immediately from the triangle inequality applied to (2.8) that the universal norm (in fact, any C^*-norm) satisfies $\big(\|\Gamma s_1 \Gamma\|_u, \ldots, \|\Gamma s_n \Gamma\|_u\big) \in B$. Moreover, from Lemma 2.2.2, the L^1-norm satisfies

$$\|\Gamma s_i \Gamma\|_{L^1}^2 \;=\; \|(\Gamma s_i \Gamma)^* * \Gamma s_i \Gamma\|_{L^1} \;=\; \sum_{j=1}^{n} \lambda_{ij}\, \|\Gamma s_j \Gamma\|_{L^1}\,.$$

Thus, $\big(\|\Gamma s_1 \Gamma\|_{L^1}, \ldots, \|\Gamma s_n \Gamma\|_{L^1}\big) \in C$. For ease of reading we will denote by $\mathbf{z} := (z_1, \ldots, z_n)$ the point $\big(\|\Gamma s_1 \Gamma\|_{L^1}, \ldots, \|\Gamma s_n \Gamma\|_{L^1}\big)$. The idea for the remaining part of the proof is to argue that $\mathbf{z} \in C$ is the point with the largest coordinates in the whole set B.

For each $1 \leq i \leq n$ let $g_i : (\mathbb{R}_0^+)^n \to \mathbb{R}$ be the function

$$g_i(x_1, \ldots, x_n) := x_i^2 - \sum_{j=1}^{n} \lambda_{ij} x_j\,.$$

The tangent hyperplane to the graph of g_i at the point $(z_1, \ldots, z_n)$ is given by the equation

$$(2z_i - \lambda_{ii})(x_i - z_i) - \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda_{ij}(x_j - z_j) = 0\,,$$

which, using the fact that $(z_1, \ldots, z_n)$ is a zero of g_i, we can reduce to

$$(2z_i - \lambda_{ii})x_i - \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda_{ij}\, x_j = z_i^2\,. \tag{2.9}$$

We claim that $2z_i - \lambda_{ii} > 0$. To see this we notice that

$$\begin{aligned} 2z_i - \lambda_{ii} \;&=\; 2\|\Gamma s_i \Gamma\|_{L^1} - \frac{L(s_i) C_{s_i^{-1}, s_i}(s_i)}{L(s_i)} \;=\; 2L(s_i) - C_{s_i^{-1}, s_i}(s_i) \\ &\geq\; 2L(s_i) - L(s_i) \;=\; L(s_i) \;>\; 0\,. \end{aligned}$$

Let us now take $A = [a_{ij}]$ to be the $n \times n$ matrix whose entries are given by $a_{ij} := -\lambda_{ij}$ for $i \neq j$, and $a_{ii} := 2z_i - \lambda_{ii}$, thus $a_{ij} \in \mathbb{R}_0^-$ for $i \neq j$ and $a_{ii} \in \mathbb{R}^+$. We can easily see from (2.9) that $A\mathbf{z} = \mathbf{z}^2$, where $\mathbf{z}^2 = (z_1^2, \dots, z_n^2)$. Consider now the set W defined by

$$W := \left\{ \mathbf{x} \in (\mathbb{R}_0^+)^n : A\mathbf{x} \leq \mathbf{z}^2 \right\}.$$

We claim that W contains the set B. To see this, let $(y_1, \dots, y_n) \in B$. We then have

$$\begin{aligned} -z_i^2 + (2z_i - \lambda_{ii})y_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_{ij}\, y_j &= -z_i^2 + 2z_i y_i - \lambda_{ii} y_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_{ij}\, y_j \\ &= -(y_i - z_i)^2 + y_i^2 - \sum_{j=1}^{n} \lambda_{ij}\, y_j \\ &\leq y_i^2 - \sum_{j=1}^{n} \lambda_{ij}\, y_j \\ &\leq 0\,, \end{aligned}$$

which implies that

$$(2z_i - \lambda_{ii})y_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_{ij}\, y_j \leq z_i^2\,,$$

and thus $(y_1, \dots, y_n) \in W$. In other words, if $\mathbf{y} \in B$, then $A\mathbf{y} \leq \mathbf{z}^2$. We can rewrite this inequality as:

$$A\mathbf{y} \leq \mathbf{z}^2 \quad \Leftrightarrow \quad 0 \leq \mathbf{z}^2 - A\mathbf{y} \quad \Leftrightarrow \quad 0 \leq A(\mathbf{z} - \mathbf{y})\,.$$

Noting that we are under the conditions of Lemma 2.2.4, because the entries of A satisfy the required conditions and $A\mathbf{z} = \mathbf{z}^2$, we conclude that $0 \leq \mathbf{z} - \mathbf{y}$, i.e. $\mathbf{y} \leq \mathbf{z}$. Thus, we conclude that $\mathbf{z}$ has bigger coordinates than any other point in B.

As we know, we have $(\|\Gamma s_1 \Gamma\|_u, \dots, \|\Gamma s_n \Gamma\|_u) \in B$, so by the above we must have $\|\Gamma s_i \Gamma\|_u \leq z_i = \|\Gamma s_i \Gamma\|_{L^1}$ for any $1 \leq i \leq n$. Thus, in particular, $\|\Gamma s \Gamma\|_u \leq \|\Gamma s \Gamma\|_{L^1}$, for the initial double coset $\Gamma s \Gamma$. Since all double cosets generate finite co-hereditary sets we conclude that this inequality holds for any double coset $\Gamma s \Gamma$, and as we explained in the beginning of the proof, this implies that the enveloping C^*-algebra of $\mathcal{H}(G, \Gamma)$ is $C^*(L^1(G, \Gamma))$. $\square$

2.3 Methods based on commutators

The basis of our study of enveloping C^*-algebras of Hecke algebras will be Corollary 2.1.5 and Theorem 2.2.1. Our goal is to apply these results to several classes of Hecke pairs, but so far we have not given any hint on how to actually ensure that a given double coset generates a finite co-hereditary set. The objective of this section is to provide some tools, based on iterated commutators, to help us accomplish this task.

Given a group G we will denote by $[s,t]$ the commutator of $s,t \in G$, i.e.

$$[s,t] := s^{-1}t^{-1}st\,.$$

More generally, given elements $s_1,\dots,s_n \in G$ we will denote by $[s_1,\dots,s_n]$ the iterated commutator defined inductively by

$$[s_1,\dots,s_n] := [[s_1,\dots,s_{n-1}],s_n]\,.$$

Let us now return to Hecke pairs (G,Γ). We will be mostly interested in commutators of the form $[g,\gamma_1,\dots,\gamma_n]$, where $g \in G$ and $\gamma_1,\dots,\gamma_n \in \Gamma$, and the reason for that is given in the following result. We recall that $S^n(X)$ stands for the n-th successor set of a set of vertices X of a directed graph, as defined in (1.25).

Proposition 2.3.1. *Let (G,Γ) be a Hecke pair and $g \in G$. Let $\{\Gamma x_n \Gamma\}_{n\in\mathbb{N}_0}$ be a sequence of double cosets satisfying the properties:*

i) $\Gamma x_0 \Gamma = \Gamma g \Gamma$,

ii) $\Gamma x_{n+1}\Gamma$ *is a successor of* $\Gamma x_n \Gamma$*, for all* $n \geq 0$.

Then, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Gamma$ such that

$$\Gamma x_n \Gamma = \Gamma[g,\gamma_1,\dots,\gamma_n]\Gamma\,,$$

for all $n \geq 1$. In particular, all elements in $S^n(\Gamma g \Gamma)$ have a representative of the form $\Gamma[g,\gamma_1,\dots,\gamma_n]\Gamma$, for some $\gamma_1,\dots,\gamma_n \in \Gamma$.

Proof: We will choose such a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ inductively on $n \in \mathbb{N}$. Suppose $n = 1$. Since $\Gamma x_1 \Gamma$ is a successor of $\Gamma g \Gamma$, it must be of the form $\Gamma x_1 \Gamma = \Gamma g^{-1}\gamma g \Gamma$ for some $\gamma \in \Gamma$. Now we notice that

$$g^{-1}\gamma g = g^{-1}\gamma g \gamma^{-1}\gamma = [g,\gamma^{-1}]\gamma\,.$$

Hence, we have

$$\Gamma x_1 \Gamma = \Gamma g^{-1} \gamma g \Gamma = \Gamma [g, \gamma^{-1}] \gamma \Gamma = \Gamma [g, \gamma^{-1}] \Gamma \,.$$

Choosing $\gamma_1 := \gamma^{-1}$ yields the desired result.

Now let us suppose that there exist elements $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\Gamma x_k \Gamma = \Gamma[g, \gamma_1, \dots, \gamma_k]\Gamma$, for every $1 \leq k \leq n$. Then, since $\Gamma x_{n+1} \Gamma$ is a successor of $\Gamma x_n \Gamma = \Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma$, we can write

$$\Gamma x_{n+1} \Gamma = \Gamma [g, \gamma_1, \dots, \gamma_n]^{-1} \gamma [g, \gamma_1, \dots, \gamma_n] \Gamma \,,$$

for some $\gamma \in \Gamma$. We have

$$\begin{aligned} \Gamma x_{n+1} \Gamma &= \Gamma [g, \gamma_1, \dots, \gamma_n]^{-1} \gamma [g, \gamma_1, \dots, \gamma_n] \gamma^{-1} \Gamma \\ &= \Gamma [g, \gamma_1, \dots, \gamma_n, \gamma^{-1}] \Gamma \,. \end{aligned}$$

Choosing $\gamma_{n+1} := \gamma^{-1}$ yields the desired result for $\Gamma x_{n+1} \Gamma$.

Hence, since we can extend any finite sequence $\gamma_1, \dots, \gamma_n$ satisfying the stated conditions to a sequence $\gamma_1, \dots, \gamma_n, \gamma_{n+1}$ still satisfying the stated conditions, it follows that there must be an infinite sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ with the desired requirements. $\square$

We will now establish a sufficient condition to ensure the finiteness of the co-hereditary set generated by an element $\Gamma g \Gamma$ based on the iterated commutators we considered above:

Theorem 2.3.2. *Let (G, Γ) be a Hecke pair and $g \in G$. Suppose that for any sequence of elements $\{\gamma_k\}_{k \in \mathbb{N}} \subseteq \Gamma$ the total number of double cosets*

$$\#\big\{ \Gamma [g, \gamma_1, \dots, \gamma_n] \Gamma : n \in \mathbb{N} \big\}$$

is finite. Then $\Gamma g \Gamma$ generates a finite co-hereditary set.

Proof: Suppose the co-hereditary set generated by $\Gamma g \Gamma$ is infinite. Then, there must exist a sequence $\{\Gamma x_n \Gamma\}_{n \in \mathbb{N}_0}$ such that

i) $\Gamma x_0 \Gamma = \Gamma g \Gamma$,

ii) $\Gamma x_{n+1} \Gamma$ is a successor of $\Gamma x_n \Gamma$, for all $n \geq 0$.

iii) $\Gamma x_{n+1} \Gamma \notin \bigcup_{i=0}^{n} S^i(\Gamma g \Gamma)$, for all $n \geq 0$.

In particular, we have that $\Gamma x_i \Gamma \neq \Gamma x_j \Gamma$ for $i \neq j$, implying that the set $\{\Gamma x_n \Gamma : n \in \mathbb{N}\}$ is infinite.

By Proposition 2.3.1 there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Gamma$ such that $\Gamma x_n \Gamma = \Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma$ for all $n \geq 1$. But, by assumption, the number of double cosets in $\{\Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma : n \in \mathbb{N}\}$ is finite. Thus we arrive at a contradiction and therefore the co-hereditary set generated by $\Gamma g \Gamma$ must be finite. $\square$

Corollary 2.3.3. *Let (G, Γ) be a Hecke pair and $g \in G$. If one of the following conditions holds, then $\Gamma g \Gamma$ generates a finite co-hereditary set:*

a) For every sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Gamma$ there exists a finite set $F \subseteq G$ and $N_0 \in \mathbb{N}$ such that $[g, \gamma_1, \dots, \gamma_k] \in F$ for all $k \geq N_0$.

b) For every sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Gamma$ there exists a number $N \in \mathbb{N}$ such that $[g, \gamma_1, \dots, \gamma_N] \in \Gamma$.

Proof: The result follows directly from Theorem 2.3.2. For a) we notice that we can write $\{\Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma : n \in \mathbb{N}\}$ as the union of the two finite sets $\{\Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma : n < N_0\}$ and $\{\Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma : n \geq N_0\}$.

For b), one can easily show, by induction, that $[g, \gamma_1, \dots, \gamma_n] \in \Gamma$, for any $n \geq N$. Thus, we have

$$\left\{\Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma : n \in \mathbb{N}\right\} = \left\{\Gamma[g, \gamma_1, \dots, \gamma_n]\Gamma : n \leq N\right\},$$

which is a finite set. $\square$

There are different classes of Hecke pairs that satisfy conditions a) and b) of the above corollary. As we shall see in more detail in the next section, condition a) is satisfied by groups satisfying certain generalized nilpotency properties, whereas b) is satisfied when Γ is a subnormal subgroup of G, for example.

2.4 Classes of Hecke Pairs

We will now use the methods developed in the previous sections to study the existence of enveloping C^*-algebras for several classes of Hecke algebras. Many of the well known results about the existence of a full Hecke C^*-algebra for some classes of Hecke pairs will be recovered in a unified approach and some new classes will also be described. The isomorphism $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$ will also be established in many of the considered classes.

It should also be noted that all the classes of Hecke algebras considered here are in fact BG^*-algebras, since our methods can be traced back to Corollary 2.1.5, but since the focus is mostly on the existence of $C^*(G,\Gamma)$ we will not mention this in every case.

This section is organized as follows: the classes of Hecke pairs from 2.4.1 to 2.4.4 have been studied in the operator algebraic literature and results about the corresponding full Hecke C^*-algebras are known. The results about the remaining classes, 2.4.5 to 2.4.12, are essentially new, with the results for the classes 2.4.5, 2.4.6 and 2.4.7 generalizing known results in the literature.

The classes we consider are presumably all different (in the sense of containment), with the notable exceptions of 2.4.5 which is a particular case of 2.4.6, and 2.4.1 which is a particular case of 2.4.7.

We would like to remark that the results discussed in this section illustrate how our methods apply for natural classes of Hecke pairs and that we have not, by any means, exhausted all the possible classes of Hecke pairs one can study through these methods.

2.4.1 Γ has Finite Index in G

When Γ has finite index in G, the pair (G,Γ) is automatically a Hecke pair, and the Hecke *-algebra is finite dimensional (actually, $\mathcal{H}(G,\Gamma)$ is finite dimensional if and only if Γ has finite index in G). As we have seen in Example 2.1.6, the co-hereditary set generated by a double coset is finite because the graph of $\mathcal{H}(G,\Gamma)$ is itself finite. Hence, Theorem 2.2.1 tells us that $C^*(G,\Gamma)$ exists and $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$.

Of course this example, investigated by Hall [18, Section 4.2], is well-known and completely understood, because a finite dimensional *-algebra is automatically complete for any *-algebra norm. Hence we necessarily have

$$C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C^*_r(G,\Gamma)\,,$$

and all these C^*-algebras are isomorphic to $\mathcal{H}(G,\Gamma)$, without having to invoke our Theorem 2.2.1.

2.4.2 (G, Γ) is Directed

Recall that (G, Γ) is said to be *directed* if $G = T^{-1}T$, where

$$T := \{t \in G : \Gamma \subseteq t\Gamma t^{-1}\}\,.$$

Directed Hecke pairs have been widely studied in the literature ([7], [18], [31], [29], [4], [24], for example), in particular because of their association with the theory of semigroup C^*-crossed products. It is known that when (G, Γ) is directed the Hecke algebra has an enveloping C^*-algebra and moreover one has

$$C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p\,,$$

(see, for example, [24, Theorem 6.4]).

With our methods we can show that $C^*(G, \Gamma)$ exists, since the Hecke algebra is in fact generated by finite co-hereditary sets. To see this, we first notice that, for $t \in T$, we have $\Gamma t \Gamma = t\Gamma$. Hence, we also have

$$(\Gamma s \Gamma)^* * \Gamma t \Gamma = \Gamma s^{-1} t \Gamma \tag{2.10}$$

for every $s, t \in T$, which means that the Hecke *-algebra is generated by the set of double cosets $\{\Gamma t \Gamma : t \in T\}$. Taking $s = t$ in equality (2.10) we see that

$$(\Gamma t \Gamma)^* * \Gamma t \Gamma = \Gamma$$

Thus, the only successor of the double coset $\Gamma t \Gamma$ is Γ. Since Γ is the only successor of itself, it follows that the co-hereditary set generated by $\Gamma t \Gamma$ has only two elements, $\Gamma t \Gamma$ and Γ, and is therefore finite. We conclude that $\mathcal{H}(G, \Gamma)$ is generated by finite co-hereditary sets and therefore $C^*(G, \Gamma)$ exists by Corollary 2.1.5.

2.4.3 Iwahori Hecke Algebras

Let (G, Γ) be a Hecke pair such that $\mathcal{H}(G, \Gamma)$ is an Iwahori Hecke algebra (see [18, Definition 5.12] for a precise definition of this concept). Sets of generators and relations have been given for this class of Hecke algebras, but for our purposes we will only need to know that:

1. There is a set $S \subseteq G$ of elements of order two such that $\mathcal{H}(G, \Gamma)$ is generated (as a *-algebra) by Γ and the double cosets $\Gamma s \Gamma$, with $s \in S$.

2. for every $s \in S$ the following relation holds:

$$(\Gamma s \Gamma)^2 = L(s)\Gamma + (L(s) - 1)\Gamma s \Gamma \,.$$

For the remaining relations in $\mathcal{H}(G, \Gamma)$, which we will not make any use in this work, we refer the reader to Hall's thesis [18, Section 5.3.1].

It was proven by Hall [18, Proposition 2.24], through an estimate on the spectral radius of certain elements, that an Iwahori Hecke algebra has an enveloping C^*-algebra (actually Hall proved this for the case $(SL_n(\mathbb{Q}_p), B)$, with $B \subseteq SL_n(\mathbb{Q}_p)$ an Iwahori subgroup, but her proof is completely general).

We can also conclude this from our methods, by proving that $\mathcal{H}(G, \Gamma)$ is generated by finite co-hereditary sets. By point 1) we only need to see that each double coset $\Gamma s \Gamma$ with $s \in S$ generates a finite co-hereditary set. So let $\Gamma s \Gamma \in \mathcal{H}(G, \Gamma)$ with $s \in S$. Since s has order two we see that $\Gamma s \Gamma$ is self-adjoint and therefore relation 2) can be rewritten as

$$(\Gamma s \Gamma)^* * \Gamma s \Gamma = L(s)\Gamma + (L(s) - 1)\Gamma s \Gamma$$

Hence, the successors of $\Gamma s \Gamma$ are only Γ and $\Gamma s \Gamma$ itself. Thus, the co-hereditary set generated by $\Gamma s \Gamma$ has only two elements, Γ and $\Gamma s \Gamma$, and is therefore finite. We conclude that $\mathcal{H}(G, \Gamma)$ is generated by finite co-hereditary sets, and is therefore a BG^*-algebra and has an enveloping C^*-algebra.

Remark 2.4.1. By a result of Hall [18, Theorem 6.10] and a result of Kaliszewski, Landstad and Quigg [24, Corollary 5.11] it is known that, for $G = SL_2(\mathbb{Q}_p)$ and Γ an Iwahori subgroup, we necessarily have

$$C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p \,.$$

The analogous result for $SL_n(\mathbb{Q}_p)$ with $n \geq 3$ is still open, as far as we know.

2.4.4 Γ is a Protonormal Subgroup of G

We recall that Γ is a *protonormal* subgroup of G (in the sense of Exel [12]), if for every $s \in G$ we have

$$\Gamma s^{-1} \Gamma s = s^{-1} \Gamma s \Gamma \,.$$

Subgroups with this property are also called *conjugate permutable subgroups* in the literature.

It was proven by Exel ([12, Proposition 12.1]) that when Γ is a protonormal subgroup of G the enveloping C^*-algebra $C^*(G,\Gamma)$ exists. Moreover, it is completely clear from his proof that $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$, since the bound he uses for the universal norm is actually the L^1-norm. Our methods can also recover this result, because in fact any double coset $\Gamma g \Gamma$ generates a finite co-hereditary set. We will actually prove that the co-hereditary set generated by $\Gamma g \Gamma$ consists only of $\Gamma g \Gamma$ and $S(\Gamma g \Gamma)$ and is therefore finite. In other words, we will prove that

$$S^n(\Gamma g \Gamma) \subseteq S(\Gamma g \Gamma)\,,$$

for every $n \in \mathbb{N}$. It suffices to prove that $S^2(\Gamma g \Gamma) \subseteq S(\Gamma g \Gamma)$. The elements of $S^2(\Gamma g \Gamma)$ are of the form $\Gamma[g,\gamma_1,\gamma_2]\Gamma$, where $\gamma_1,\gamma_2 \in \Gamma$, by Proposition 2.3.1. We have that

$$\begin{aligned}[g,\gamma_1,\gamma_2] &= [g,\gamma_1]^{-1}\gamma_2^{-1}[g,\gamma_1]\gamma_2 \\ &= \gamma_1^{-1}g^{-1}(\gamma_1 g \gamma_2^{-1} g^{-1})\gamma_1^{-1} g \gamma_1 \gamma_2\,.\end{aligned}$$

Since Γ is a protonormal subgroup there exist $\theta,\omega \in \Gamma$ such that $\gamma_1 g \gamma_2^{-1} g^{-1} = g\theta g^{-1}\omega$. Thus, we get

$$\begin{aligned}[g,\gamma_1,\gamma_2] &= \gamma_1^{-1}g^{-1}(g\theta g^{-1}\omega)\gamma_1^{-1} g \gamma_1 \gamma_2 \\ &= \gamma_1^{-1}\theta g^{-1}\omega\gamma_1^{-1} g \gamma_1 \gamma_2\,,\end{aligned}$$

and therefore

$$\begin{aligned}\Gamma[g,\gamma_1,\gamma_2]\Gamma &= \Gamma\gamma_1^{-1}\theta g^{-1}\omega\gamma_1^{-1} g \gamma_1 \gamma_2\Gamma \\ &= \Gamma g^{-1}\omega\gamma_1^{-1} g\Gamma\,.\end{aligned}$$

By Remark 1.2.7, $\Gamma g^{-1}\omega\gamma_1^{-1} g\Gamma \in S(\Gamma g \Gamma)$. This finishes the proof.

2.4.5 Γ is Subnormal in G

Hecke pairs (G,Γ) in which Γ is normal in a normal subgroup of G have been widely studied in the literature, in particular when G is a semi-direct product ([7], [31], [29], [24]), and it is known that in this case $\mathcal{H}(G,\Gamma)$ has an enveloping C^*-algebra and moreover

$$C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p\,,$$

(see, for example, [24, Theorem 5.13]).

We are now going to prove that when Γ is a subnormal subgroup of G, $C^*(G,\Gamma)$ exists and $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$. Recall that Γ is *subnormal* in G if there are subgroups $H_0, H_1, \dots, H_n$ such that

$$\Gamma = H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_0 = G\,,$$

where the notation $H_{i+1} \trianglelefteq H_i$ means that H_{i+1} is a normal subgroup of H_i.

We claim that when Γ is subnormal in G, all double cosets $\Gamma s \Gamma$ generate finite co-hereditary sets. To see this we will use Corollary 2.3.3. Let $s \in G$ and $\{\gamma_k\}_{k\in\mathbb{N}} \subseteq \Gamma$. We will prove by induction that $[s, \gamma_1, \dots, \gamma_k] \in H_k$ for $1 \leq k \leq n$. For $k = 1$ this follows from the following observation:

$$[s,\gamma_1] = s^{-1}\gamma_1^{-1}s\gamma_1 \in s^{-1}\Gamma s\gamma_1 \subseteq s^{-1}H_1 s\gamma_1 = H_1\gamma_1 = H_1\,.$$

Now, let us prove that $k \Rightarrow k+1$. For simplicity, let us write $x_k := [s, \gamma_1, \dots, \gamma_k]$, which by induction hypothesis is an element of H_k. Thus, we have

$$\begin{array}{rcl} [s,\gamma_1,\dots,\gamma_k,\gamma_{k+1}] & = & [x_k,\gamma_{k+1}] \;\in\; x_k^{-1}\Gamma x_k\gamma_{k+1} \\ & \subseteq & x_k^{-1}H_{k+1}x_k\gamma_{k+1} \;=\; H_{k+1}\gamma_{k+1} \\ & = & H_{k+1}\,. \end{array}$$

Thus, for any sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ we have $[s, \gamma_1, \dots, \gamma_n] \in \Gamma$, which by Corollary 2.3.3 *b*) implies that $\Gamma s\Gamma$ generates a finite co-hereditary set. Since this is true for all double cosets, Theorem 2.2.1 tells us that $C^*(G,\Gamma)$ exists and $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$.

Remark 2.4.2. It is known that any subgroup Γ of a nilpotent group G is necessarily a subnormal subgroup (see, for example, [28, §62]). Hence already from this we can conclude that the Hecke algebra of any Hecke pair (G,Γ), with G a nilpotent group, has an enveloping C^*-algebra (which coincides with $C^*(L^1(G,\Gamma))$). In fact, this holds for any group G whose subgroups are all subnormal. Groups with this property form a class that strictly contains the class of nilpotent groups ([39, Theorem 6.11]). We will prove similar results for other classes of groups which strictly generalize the class of nilpotent groups.

Example 2.4.3. Let G be the group of $n \times n$ upper triangular matrices with 1's on the diagonal and with entries in $\mathbb{Q}$ and let Γ be the subgroup of those matrices with entries in $\mathbb{Z}$. It can be checked, although we will not do so here, that (G, Γ) forms a Hecke pair. The subgroup Γ is subnormal with

$$\Gamma = H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_1 = G \,,$$

where H_k is the subgroup of matrices in G whose first $k-1$ upper diagonals have entries in $\mathbb{Z}$. The group G is nilpotent and its 3×3 version is the rational Heisenberg group discussed in [24, Example 10.7].

2.4.6 Γ is Ascendant in G

Recall that Γ is said to be *ascendant* in G if there is a normal series $\{H_i\}_{i \in \mathbb{N}_0}$,

$$\Gamma = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_i \trianglelefteq \ldots$$

that ends in the group G, in the sense that $\bigcup_{i \in \mathbb{N}_0} H_i = G$. Of course, the series is finite precisely when Γ is subnormal in G.

We will now prove that if Γ is ascendant in G, then every double coset generates a finite co-hereditary set, therefore implying that $C^*(G, \Gamma)$ exists and is isomorphic to $C^*(L^1(G, \Gamma))$.

Let $\Gamma s \Gamma$ be any double coset in $\mathcal{H}(G, \Gamma)$, with representative $s \in G$. Since Γ is ascendant, s must belong to one of the subgroups H_n, with $n \in \mathbb{N}_0$. Of course, Γ is a subnormal subgroup of H_n, and as we saw in the subnormal case, this implies that the co-hereditary set generated by $\Gamma s \Gamma$ is necessarily finite.

2.4.7 Γ has Finitely Many Conjugates in G

Suppose Γ has finitely many conjugates in G, or equivalently, the normalizer of Γ has finite index in G. Then, $C^*(G, \Gamma)$ exists and $C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma))$ because any double coset generates a finite co-hereditary set. To see this, let $\Gamma g \Gamma$ be a double coset and let $g_1^{-1} \Gamma g_1, \ldots, g_n^{-1} \Gamma g_n$ be the conjugates of Γ. With the possible exception of $\Gamma g \Gamma$ itself, any element in the co-hereditary set generated by $\Gamma g \Gamma$ is a successor of another element. Hence, by Remark 1.2.7, any such element is of the form

$$\Gamma x^{-1} \gamma x \Gamma \,,$$

where $x \in G$ and $\gamma \in \Gamma$. We can then write $x^{-1} \gamma x = g_i^{-1} \theta g_i$, for some $i \in \{1, \ldots, n\}$ and $\theta \in \Gamma$, and therefore $\Gamma x^{-1} \gamma x \Gamma = \Gamma g_i^{-1} \theta g_i \Gamma$. Thus, apart possibly from $\Gamma g\Gamma$, all elements in the co-hereditary set generated by $\Gamma g\Gamma$ are successors of some $\Gamma g_i\Gamma$, with $1 \leq i \leq n$, by Remark 1.2.7 again. Thus, this co-hereditary set must be finite.

2.4.8 G is Finite-by-Nilpotent

Recall that a group G is called *nilpotent* if its lower central series stabilizes at $\{e\}$ after finitely many steps, i.e. if the normal series defined inductively by

$$G_0 := G\,, \qquad\qquad G_{n+1} := [G_n, G]\,,$$

is such that $G_k = \{e\}$, for some $k \in \mathbb{N}$.

Recall also that a group G is said to be *finite-by-nilpotent* if G has a finite normal subgroup K such that G/K is nilpotent, i.e. if G is an extension of a finite group by a nilpotent group. In particular, all nilpotent groups are finite-by-nilpotent (taking $K = \{e\}$). Moreover, the class of finite-by-nilpotent groups is strictly larger than the class of nilpotent groups, as every finite group belongs to the former class but not to the latter.

Finite-by-nilpotent groups also admit a nice description in terms of their lower central series: it is known that finite-by-nilpotent groups are precisely those whose lower central series stabilizes at a finite group.

We are now going to show that for any Hecke pair (G, Γ) where G is finite-by-nilpotent, every double coset $\Gamma s\Gamma$ generates a finite co-hereditary set, implying that $C^*(G, \Gamma)$ exists and coincides with $C^*(L^1(G, \Gamma))$.

Let $s \in G$ and $\{\gamma_k\}_{k\in\mathbb{N}} \subseteq \Gamma$. It is clear that $[s, \gamma_1, \ldots, \gamma_k] \in G_k$. Since the series $\{G_k\}$ eventually stabilizes at a finite subgroup, it follows directly from Corollary 2.3.3 a) that $\Gamma s\Gamma$ generates a finite co-hereditary set. This concludes the proof.

2.4.9 G is Hypercentral

Recall that a group G is said to be a *hypercentral group* (also called a ZA-*group*) if its upper central series, possibly continued transfinitely, stabilizes at the whole group G. For a rigorous definition of this concept, we refer the reader to [38, section 12.2] for example. Another characterization of hypercentral groups, which is the one we will use, is given by the following result:

Theorem 2.4.4 (Lemma, page 219, §63, [28]). *A group G is hypercentral if and only if it satisfies the following property: for any $s \in G$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subset G$ there is a $k \in \mathbb{N}$ such that*

$$[s, x_1, \dots, x_k] = e\,.$$

We will now prove that if (G, Γ) is a Hecke pair with G a hypercentral group, then every double coset $\Gamma s \Gamma$ generates a finite co-hereditary set, so that $C^*(G, \Gamma)$ exists and $C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma))$. This is a direct application of Corollary 2.3.3 *a*), taking $F = \{e\}$, given the characterization of hypercentral groups of Theorem 2.4.4.

Remark 2.4.5. The class of hypercentral groups also strictly contains the class of nilpotent groups (see Example 2.4.6), and moreover it is known that every hypercentral group is locally nilpotent (but not vice-versa). Thus, we have found another class of groups G, satisfying a nilpotent-type property, for which the Hecke algebra $\mathcal{H}(G, \Gamma)$ of any Hecke pair (G, Γ) has an enveloping C^*-algebra (which coincides with $C^*(L^1(G, \Gamma))$).

Example 2.4.6. Let $\mathbb{Z}_{2^\infty}$ be the 2-quasicyclic group, i.e. the group of all the 2^n-th roots of unity for all $n \in \mathbb{N}$. This group is the Pontryagin dual of the group of 2-adic integers. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}_{2^\infty}$ by mapping an element to its inverse. The generalized dihedral group

$$G := \mathbb{Z}_{2^\infty} \rtimes \big(\mathbb{Z}/2\mathbb{Z}\big)$$

is a group which is hypercentral, but not nilpotent.

2.4.10 G is an FC-group and Γ is Finite

Recall that a group G is said to be FC if every element s has finitely many conjugates, i.e. the set $\mathcal{C}_s := \{t^{-1}st : t \in G\}$ is finite. It can be seen that every subgroup $\Gamma \subseteq G$ of an FC-group is a Hecke subgroup, because

$$\Gamma s \Gamma = \bigcup_{\gamma\in\Gamma} \gamma s \Gamma = \bigcup_{\gamma\in\Gamma} \gamma s \gamma^{-1} \Gamma \subseteq \bigcup_{x\in\mathcal{C}_s} x\Gamma\,,$$

and the last union is finite.

FC groups are a generalization of both finite and abelian groups, and share many common properties with these classes. They were extensively studied by B. H. Neumann and others, starting with the article [33]. The analogous class of groups in the locally compact setting (groups in which the conjugacy class of any element has compact closure) is usually denoted by FC^- and has also been widely studied, since it is a direct generalization of both compact and abelian locally compact groups (see [35, Chapter 12] for an account).

When G is a FC-group and $\Gamma \subseteq G$ is a finite subgroup, we can prove that every double coset $\Gamma s \Gamma$ generates a finite co-hereditary set, so that $C^*(G, \Gamma)$ exists and $C^*(G, \Gamma) \cong C^*(L^1(G, \Gamma))$. To see this, let $s \in G$ and $\{\gamma_k\}_{k \in \mathbb{N}} \subseteq \Gamma$. Also, let $\Gamma = \{\theta_1, \dots, \theta_n\}$ and for each $1 \leq i \leq n$ let us denote by $S_i \subseteq \mathbb{N}$ the set

$$S_i := \{j \in \mathbb{N} : \gamma_j = \theta_i\} .$$

Of course, the sets S_i are mutually disjoint and their union is $\mathbb{N}$. We have that

$$\begin{aligned} \{\Gamma[s, \gamma_1, \dots, \gamma_k]\Gamma : k \in \mathbb{N}\} &= \bigcup_{i=1}^{n} \{\Gamma[s, \gamma_1, \dots, \gamma_k]\Gamma : k \in S_i\} \\ &= \bigcup_{i=1}^{n} \{\Gamma[s, \gamma_1, \dots, \gamma_{k-1}, \theta_i]\Gamma : k \in S_i\} . \end{aligned}$$

Now we notice that

$$\Gamma[s, \gamma_1, \dots, \gamma_{k-1}, \theta_i]\Gamma = \Gamma[s, \gamma_1, \dots, \gamma_{k-1}]^{-1} \theta_i^{-1} [s, \gamma_1, \dots, \gamma_{k-1}]\Gamma .$$

Since there are only finitely many conjugates of θ_i^{-1}, it follows that the set $\{\Gamma[s, \gamma_1, \dots, \gamma_{k-1}, \theta_i]\Gamma : k \in S_i\}$ is finite, and therefore $\{\Gamma[s, \gamma_1, \dots, \gamma_k]\Gamma : k \in \mathbb{N}\}$ is finite. Thus, by Theorem 2.3.2, the co-hereditary set generated by $\Gamma s \Gamma$ is finite.

2.4.11 G is Locally-Nilpotent and Γ is Finite

Recall that a group G is said to be *locally-nilpotent* if every finitely generated subgroup of G is nilpotent.

Let G be a locally-nilpotent group and Γ a finite subgroup. The pair (G, Γ) is automatically a Hecke pair since Γ is finite. We are now going to prove that each double coset $\Gamma s \Gamma$ generates a finite co-hereditary set, implying that $C^*(G,\Gamma)$ exists and coincides with $C^*(L^1(G,\Gamma))$. To see this, let $\langle s,\Gamma\rangle \subseteq G$ be the subgroup generated by s and Γ. This subgroup is finitely generated, hence nilpotent. Thus, as we have proven above, $\Gamma s\Gamma \in \mathcal{H}(\langle s,\Gamma\rangle,\Gamma) \subseteq \mathcal{H}(G,\Gamma)$ generates a finite co-hereditary set.

2.4.12 G is Locally-Finite and Γ is Finite

Recall that a group G is said to be *locally-finite* if every finitely generated subgroup of G is finite.

Let G be a locally-finite group and Γ a finite subgroup. The pair (G,Γ) is automatically a Hecke pair since Γ is finite. We are now going to prove that each double coset $\Gamma s\Gamma$ generates a finite co-hereditary set, implying that $C^*(G,\Gamma)$ exists and coincides with $C^*(L^1(G,\Gamma))$. To see this, let $\langle s,\Gamma\rangle \subseteq G$ be the subgroup generated by s and Γ. This subgroup is finitely generated, hence finite. Thus, as we have proven above, $\Gamma s\Gamma \in \mathcal{H}(\langle s,\Gamma\rangle,\Gamma) \subseteq \mathcal{H}(G,\Gamma)$ generates a finite co-hereditary set.

An interesting feature of Hecke pairs arising from locally finite groups is that they give rise to AF Hecke algebras. In that regard we have the following result:

Proposition 2.4.7. *Let (G,Γ) be a Hecke pair where G is countable and Γ is a finite subgroup. Then $\mathcal{H}(G,\Gamma)$ is an AF *-algebra if and only if G is locally finite.*

Proof: $(\Longleftarrow)$ Assume G is locally finite. Since G is assumed countable, let us fix an enumeration of its elements $G=\{g_1,g_2,\dots\}$ and for each $n\in\mathbb{N}$ let us define H_n as the subgroup $H_n := \langle \Gamma, g_1,\dots,g_n\rangle$. It is clear that $\{H_n\}_{n\in\mathbb{N}}$ forms an increasing sequence of finitely generated subgroups, such that $\bigcup H_n = G$. Moreover, since G is locally finite, each H_n is a finite group which contains Γ. Hence, we have a sequence of finite dimensional Hecke algebras $\{\mathcal{H}(H_n,\Gamma)\}_{n\in\mathbb{N}} \subseteq \mathcal{H}(G,\Gamma)$ satisfying $\bigcup \mathcal{H}(H_n,\Gamma) = \mathcal{H}(G,\Gamma)$. Thus, $\mathcal{H}(G,\Gamma)$ is an *AF* *-algebra.

$(\Longrightarrow)$ Assume that $\mathcal{H}(G,\Gamma)$ is an AF *-algebra. Then any element $f\in\mathcal{H}(G,\Gamma)$ lies in a finite dimensional *-subalgebra, and is therefore algebraic over $\mathbb{C}$. It then follows from [26, Proposition 2.6] that G is locally finite. $\square$

Example 2.4.8. Similarly to Example 2.4.6, let p be a prime number and $\mathbb{Z}_{p^\infty}$ be the p-quasicyclic group (which is the Pontryagin dual of the group of

p-adic integers). The generalized dihedral group

$$G := \mathbb{Z}_{p^\infty} \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

is locally finite (but not locally nilpotent unless $p = 2$).

Chapter 3

On the completions $C^*(L^1(G,\Gamma))$ and $pC^*(\overline{G})p$

In this chapter we address the question of when do the two C^*-completions of the Hecke algebra, $C^*(L^1(G,\Gamma))$ and $pC^*(\overline{G})p$, coincide. A result of Kaliszewski, Landstad and Quigg's in [24] shows that this is the case whenever the Schlichting completion $\overline{G}$ is a Hermitian group. We will generalize their result to the case when $\overline{G}$ has a quasi-symmetric group algebra (a notion we will introduce below). Our generalization is such that it includes the class of all Hecke pairs (G,Γ) for which G (or G_r or $\overline{G}$) has subexponential growth.

As a consequence of this result, combined together with the results on Chapter 2 and Tzanev's theorem, we will show in Section 3.4 that

$$C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C^*_r(G,\Gamma)\,,$$

for several classes of Hecke pairs, including all Hecke pairs (G,Γ) where G is a nilpotent group. In this way we prove that Hall's equivalence holds for all such classes of Hecke pairs.

We will also show, in Section 3.5, that there are Hecke pairs for which $C^*(L^1(G,\Gamma)) \ncong pC^*(\overline{G})p$, with $(SL_2(\mathbb{Q}_q), SL_2(\mathbb{Z}_q))$ being one such example.

3.1 Quasi-symmetric group algebras

Given a *-algebra A and an element $a \in A$ we will use throughout this chapter the notations $\sigma_A(a)$ to denote the spectrum of a relative to A, and $R_A(a)$ to denote the spectral radius of a relative to A.

Recall, for example from [35], that a *-algebra A is said to be:

- *Hermitian* if $\sigma_A(a) \subseteq \mathbb{R}$, for any self-adjoint element $a = a^*$ of A.

- *symmetric* if $\sigma_A(a^*a) \subseteq \mathbb{R}_0^+$, for any $a \in A$.

It is an easy fact that symmetry implies Hermitianess, since $\sigma_A(a)^2 = \sigma_A(a^2)$. The two properties are equivalent for Banach *-algebras, as asserted by the Shiralli-Ford theorem [40].

Recall also that a locally compact group G is called *Hermitian* if $L^1(G)$ is a Hermitian (equivalently, symmetric) Banach *-algebra. The class of Hermitian groups satisfies some known closure properties, some of which we list below:

1. The class of Hermitian groups is closed under taking open subgroups and quotients [35, Theorem 12.5.18].

2. Let $1 \to H \to G \to G/H \to 1$ be an extension of locally compact groups. If H is Hermitian and G/H is finite, then G is Hermitian [35, Theorem 12.5.18].

The class of groups we are interested in this chapter arise by relaxing the condition of symmetry on the group algebra:

Definition 3.1.1. Let G be a locally compact group. We will say that the group algebra $L^1(G)$ is *quasi-symmetric* if $\sigma_{L^1(G)}(f^* * f) \subseteq \mathbb{R}_0^+$ for any compactly supported continuous function f.

Clearly, Hermitian groups have a quasi-symmetric group algebra. Another important class of groups with this property is that of groups with subexponential growth, as was essentially discovered by Hulanicki [22], [21]. It follows as a direct corollary of the two results below:

Theorem 3.1.2 (Hulanicki [21]). *Let G be a locally compact group with subexponential growth and $\lambda : L^1(G) \to B(L^2(G))$ its left regular representation. We have that*

$$R_{L^1(G)}(f) = \|\lambda(f)\|$$

for any self-adjoint $f = f^*$ continuous function of compact support.

The following result, in the generality presented here, is due to Barnes [3] and is based on work by Hulanicki [23].

Theorem 3.1.3 (Hulanicki, Barnes). *Let A be a Banach *-algebra and $B \subseteq A$ a *-subalgebra. Suppose that $\pi : A \to B(\mathscr{H})$ is a faithful *-representation such that*

$$R_A(b) = \|\pi(b)\|\,,$$

for all self-adjoint elements $b = b^*$ in B. Then $\sigma_A(b) = \sigma_{B(\mathscr{H})}(\pi(b))$ for every $b \in B$.

Corollary 3.1.4. *If G is a locally compact group with subexponential growth, then $L^1(G)$ is quasi-symmetric.*

Proof: We consider A and B as in the previous theorem to be $L^1(G)$ and $C_c(G)$ respectively. By Hulanicki's theorem we know that the conditions of Theorem 3.1.3 are met with respect to the left regular representation λ, and therefore $\sigma_{L^1(G)}(f^* * f) = \sigma_{B(L^2(G))}(\lambda(f^* * f))$ for any $f \in C_c(G)$. Thus, $\sigma_{L^1(G)}(f^* * f) \subseteq \mathbb{R}_0^+$ for $f \in C_c(G)$, i.e. $L^1(G)$ is quasi-symmetric. $\square$

The following result is the main result in this section and explains the reason for considering quasi-symmetric group algebras in the context of C^*-completions of Hecke pairs.

Theorem 3.1.5. *Let (G, Γ) be a Hecke pair. If $\overline{G}$ has a quasi-symmetric group algebra, then*

$$C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p\,.$$

In particular, there is a category equivalence between *-representations of $L^1(G,\Gamma)$ and unitary representations of G generated by the Γ-fixed vectors.

Lemma 3.1.6. *Let (G, Γ) be a Hecke pair and $f \in pL^1(\overline{G})p$. We have that $\sigma_{pL^1(\overline{G})p}(f) \subseteq \sigma_{L^1(\overline{G})}(f)$.*

Proof: Let us denote by $L^1(\overline{G})^\dagger$ the minimal unitization of $L^1(\overline{G})$ and let $\mathbf{1} \in L^1(\overline{G})^\dagger$ be its unit. Let $\lambda \in \mathbb{C}$ and suppose that $f - \lambda\mathbf{1}$ is invertible in $L^1(\overline{G})^\dagger$. We want to prove that $f - \lambda p$ is invertible in $pL^1(\overline{G})p$. Invertibility of $f - \lambda\mathbf{1}$ in $L^1(\overline{G})^\dagger$ means that there exist $g \in L^1(\overline{G})$ and $\beta \in \mathbb{C}$ such that $\mathbf{1} = (f - \lambda\mathbf{1})(g + \beta\mathbf{1})$. Hence we have

$$\begin{aligned} p &= p(f - \lambda\mathbf{1})(g + \beta\mathbf{1})p &&= (pf - \lambda p)(gp + \beta p) \\ &= (fp - \lambda p)(gp + \beta p) &&= (f - \lambda p)p(gp + \beta p) \\ &= (f - \lambda p)(pgp + \beta p)\,. \end{aligned}$$

Hence, $f - \lambda p$ is invertible in $pL^1(\overline{G})p$ and this finishes the proof. $\square$

Proof of Theorem 3.1.5 Due to the canonical isomorphism $L^1(G,\Gamma) \cong pL^1(\overline{G})p$, it is enough to prove that $C^*(pL^1(\overline{G})p) \cong pC^*(\overline{G})p$. By [24, Corollary 5.11] we only need to show that every representation of $pL^1(\overline{G})p$ is $\langle\rangle_R$-positive. Let $\pi : pL^1(\overline{G})p \to B(\mathscr{H})$ be a *-representation and $f \in L^1(\overline{G})p$. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C_c(\overline{G})p$ such that $g_n \to f$ in $L^1(\overline{G})$. Then, we also have $g_n^* * g_n \to f^* * f$ in $L^1(\overline{G})$. It is a standard fact that

$$\sigma_{B(\mathscr{H})}(\pi(g_n^* * g_n)) \subseteq \sigma_{pL^1(\overline{G})p}(g_n^* * g_n)\,,$$

and by Lemma 3.1.6 we have $\sigma_{pL^1(\overline{G})p}(g_n^* * g_n) \subseteq \sigma_{L^1(\overline{G})}(g_n^* * g_n)$. Moreover, since $L^1(\overline{G})$ is quasi-symmetric we have that $\sigma_{L^1(\overline{G})}(g_n^* * g_n) \subseteq \mathbb{R}_0^+$. All these inclusions combined give

$$\sigma_{B(\mathscr{H})}(\pi(g_n^* * g_n)) \subseteq \sigma_{pL^1(\overline{G})p}(g_n^* * g_n) \subseteq \sigma_{L^1(\overline{G})}(g_n^* * g_n) \subseteq \mathbb{R}_0^+\,,$$

and therefore $\pi(g_n^* * g_n)$ is a positive operator for every $n \in \mathbb{N}$. Thus, the limit $\pi(f^* * f) = \lim \pi(g_n^* * g_n)$ is also a positive operator. In other words, $\pi(\langle f, f\rangle_R) \geq 0$. $\square$

As a consequence we immediately recover Kaliszewski, Landstad and Quigg's original result and also that $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$ for Hecke pairs arising from groups of subexponential growth.

Corollary 3.1.7 ([24], Theorem 5.14). *Let (G,Γ) be a Hecke pair. If $\overline{G}$ is Hermitian, then $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$.*

Corollary 3.1.8. *Let (G,Γ) be a Hecke pair. If one of the groups G, G_r or $\overline{G}$ has subexponential growth, then $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$.*

Proof: By Corollary 1.4.5, if G or G_r has subexponential growth, then so does $\overline{G}$ in its totally disconnected locally compact topology.

If $\overline{G}$ has subexponential growth, then $L^1(\overline{G})$ is quasi-symmetric (Corollary 3.1.4), and therefore $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ by Theorem 3.1.5.

The isomorphism $pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$ follows from Tzanev's theorem (Theorem 1.2.10 in the present work), due to the fact that subexponential growth implies the amenability of the group $\overline{G}$. $\square$

3.2 A remark on subexponential growth for Hecke pairs

The hypothesis in Corollary 3.1.8 require that one of the groups G, G_r or $\overline{G}$ has subexponential growth. A natural question to ask is if there is a reasonable definition of subexponential growth for a Hecke pair (G, Γ). Such a definition should heuristically mean that the "quotient" G/Γ has subexponential growth, and could in principle be taken as the hypothesis in Corollary 3.1.8 and render a more general result. We say more general because one should expect that subexponential growth of G (or G_r or $\overline{G}$) would imply subexponential growth of the pair (G, Γ), since this property passes to quotients.

As we shall see, it is possible to give such a definition, but this turns out to be equivalent to the Schlichting completion $\overline{G}$ having subexponential growth, as it is intuitively expected: since $\overline{\Gamma}$ is compact, subexponential growth of $\overline{G}/\overline{\Gamma}$ is equivalent to subexponential growth of $\overline{G}$.

Let (G, Γ) be a Hecke pair. Given a finite subset $A \subseteq \Gamma \backslash G/\Gamma$ of double cosets, we will denote by $L(A) := \sum_{[g] \in A} L(g)$ the total number of left cosets inside A. Also, if $A, B \subseteq \Gamma \backslash G/\Gamma$ are finite subsets we will denote by $AB \subseteq \Gamma \backslash G/\Gamma$ the set

$$AB := \{[g] \in \Gamma \backslash G/\Gamma : \Gamma g \Gamma \subseteq \Gamma a \Gamma b \Gamma, \text{ for some } [a] \in A, [b] \in B\},$$

which is itself a finite set. Moreover, for $n \in \mathbb{N}$ we define A^n inductively as $A^n := AA^{n-1}$, with $A^0 := A$.

Definition 3.2.1. We will say that a Hecke pair (G, Γ) has *subexponential growth* if for every finite set $A \subseteq \Gamma \backslash G/\Gamma$ we have

$$\limsup_{n \to \infty} L(A^n)^{\frac{1}{n}} = 1 .$$

We note that when Γ is a normal subgroup Definition 3.2.1 means precisely that the quotient group G/Γ has subexponential growth.

Proposition 3.2.2. *The following statements are equivalent:*

(i) (G, Γ) *has subexponential growth.*

(ii) (G_r, Γ_r) *has subexponential growth.*

(iii) $(\overline{G}, \overline{\Gamma})$ *has subexponential growth.*

(iv) $\overline{G}$ *has subexponential growth.*

Proof: It is clear that (G,Γ), (G_r,Γ_r) and $(\overline{G},\overline{\Gamma})$ have exactly the same growth rate, since we can canonically identify the double coset spaces $\Gamma\backslash G/\Gamma$, $\Gamma_r\backslash G_r/\Gamma_r$ and $\overline{\Gamma}\backslash\overline{G}/\overline{\Gamma}$, and also the corresponding Hecke algebras $\mathcal{H}(G,\Gamma)$, $\mathcal{H}(G_r,\Gamma_r)$ and $\mathcal{H}(\overline{G},\overline{\Gamma})$. So it remains to see that *(iii)* $\Leftrightarrow$ *(iv)*.

To see that *(iii)* $\Rightarrow$ *(iv)* let us consider a compact neighbourhood $A\subseteq\overline{G}$. Since the set $\overline{\Gamma}A\overline{\Gamma}\subseteq\overline{G}$ is both compact and open, it follows that

$$B := \overline{\Gamma}\backslash A/\overline{\Gamma}\,,$$

is a finite set of double cosets, and it is not difficult to see that $\overline{\Gamma}\backslash A^n/\overline{\Gamma}\subseteq B^n$.

Considering the normalized (left) Haar measure μ on $\overline{G}$ so that $\mu(\overline{\Gamma})=1$, we have that

$$\begin{aligned} L(\overline{\Gamma}\backslash A^n/\overline{\Gamma}) &= \sum_{[g]\in\overline{\Gamma}\backslash A^n/\overline{\Gamma}} L(g) = \sum_{[g]\in\overline{\Gamma}\backslash A^n/\overline{\Gamma}} \mu(\overline{\Gamma}g\overline{\Gamma}) \\ &= \mu\Big(\bigcup_{[g]\in\overline{\Gamma}\backslash A^n/\overline{\Gamma}} \overline{\Gamma}g\overline{\Gamma}\Big) = \mu(\overline{\Gamma}A^n\overline{\Gamma})\,. \end{aligned}$$

Hence, from the fact that $A^n\subseteq\overline{\Gamma}A^n\overline{\Gamma}$ and the assumption that $(\overline{G},\overline{\Gamma})$ has subexponential growth, it follows that

$$\begin{aligned} \limsup_{n\to\infty}\mu(A^n)^{\frac{1}{n}} &\leq \limsup_{n\to\infty}\mu(\overline{\Gamma}A^n\overline{\Gamma})^{\frac{1}{n}} = \limsup_{n\to\infty}L(\overline{\Gamma}\backslash A^n/\overline{\Gamma})^{\frac{1}{n}} \\ &\leq \limsup_{n\to\infty}L(B^n)^{\frac{1}{n}} = 1\,. \end{aligned}$$

Let us now prove the direction *(iv)* $\Rightarrow$ *(iii)*. For any given set $A\subseteq\overline{\Gamma}\backslash\overline{G}/\overline{\Gamma}$ there is a correspondent set $\widetilde{A}\subseteq\overline{G}$, consisting of the union of all the double cosets in A, i.e. $\widetilde{A}:=\{g\in\overline{G}:[g]\in A\}$. It is not difficult to see that $\widetilde{AB}=\widetilde{A}\widetilde{B}$, for any $A,B\in\overline{\Gamma}\backslash\overline{G}/\overline{\Gamma}$, and therefore $\widetilde{A^n}=\big(\widetilde{A}\big)^n$.

Let us take a finite set $A\subseteq\overline{\Gamma}\backslash\overline{G}/\overline{\Gamma}$. We have

$$\begin{aligned} \limsup_{n\to\infty}L(A^n)^{\frac{1}{n}} &= \limsup_{n\to\infty}\mu\big(\widetilde{A^n}\big)^{\frac{1}{n}} \\ &= \limsup_{n\to\infty}\mu\big(\big(\widetilde{A}\big)^n\big)^{\frac{1}{n}} \\ &= 1\,. \end{aligned}$$

$\square$

3.3 Further remarks on groups with a quasi-symmetric group algebra

The classes of Hermitian groups and groups with subexponential growth are in general different. On one side, there are examples of Hermitian groups which do not have subexponential growth, such as the affine group of the real line $\mathrm{Aff}(\mathbb{R}) := \mathbb{R} \rtimes \mathbb{R}^*$, with its usual topology as a (connected) Lie group, as shown by Leptin [32]. On the other side, there are examples of groups with subexponential growth which are not Hermitian, such as the Fountain-Ramsay-Williamson group [15], which is the discrete group with the presentation

$$\left\langle \{u_j\}_{j\in\mathbb{N}} \mid u_j^2 = e \ \text{ and } \ u_i u_j u_k u_j = u_j u_k u_j u_i \ \ \forall i, j < k \in \mathbb{N} \right\rangle .$$

Fountain, Ramsay and Williamson showed that this group is not Hermitian despite being locally finite (thus, having subexponential growth). Another such example was given by Hulanicki in [20].

Using these examples we can show that the class of groups with a quasi-symmetric group algebra is strictly larger than the union of the classes of Hermitian groups and groups with subexponential growth. In that regard we have the following result:

Proposition 3.3.1. *Let H be a Hermitian locally compact group with exponential growth and let L be a discrete locally finite group which is not Hermitian. The locally compact group $G := H \times L$ has a quasi-symmetric group algebra, but it is neither Hermitian nor has subexponential growth.*

An example of such a group is given by taking $H := \mathrm{Aff}(\mathbb{R})$ and L the Fountain-Ramsay-Williamson group.

Proof: Let us first prove that $G := H \times L$ has a quasi-symmetric group algebra. Given a function $f \in C_c(G)$, the product $f^* * f$ also has compact support, and since L is discrete, the support of $f^* * f$ must lie inside some set of the form $H \times F$, where $F \subseteq L$ is a finite set. Since L is locally finite, F generates a finite subgroup $\langle F \rangle \subseteq G$. Now $H \times \langle F \rangle$ is an open subgroup of G, so that

$$L^1(H \times \langle F \rangle) \subseteq L^1(G) .$$

The group $H \times \langle F \rangle$ is Hermitian, being a finite extension of a Hermitian group, and therefore $\sigma_{L^1(H\times\langle F\rangle)}(f^* * f) \subseteq \mathbb{R}_0^+$. This implies that

$$\sigma_{L^1(G)}(f^* * f) \subseteq \sigma_{L^1(H\times\langle F\rangle)}(f^* * f) \subseteq \mathbb{R}_0^+ ,$$

which shows that G is quasi-symmetric.

This group is not Hermitian, because it has a quotient (L) which is not Hermitian, and it does not have subexponential growth because it has a quotient (H) which does not have subexponential growth. $\square$

Since in the present work we are directly concerned with totally disconnected groups (because of the Schlichting completion), it would be interesting to know if there are examples of totally disconnected groups with a quasi-symmetric group algebra, but which are not Hermitian nor have subexponential growth. We do not know the answer to this question. The example considered in Proposition 3.3.1 is of course not totally disconnected since $\mathrm{Aff}(\mathbb{R})$ is a connected group. But in view of Proposition 3.3.1, it would suffice to answer affirmatively the following more fundamental problem:

Question 3.3.2. Is there any Hermitian, totally disconnected group, with exponential growth?

As we pointed out above, there are examples of locally compact groups (even connected ones) which are Hermitian and have exponential growth, such as $\mathrm{Aff}(\mathbb{R})$, but the question of whether this can happen in the totally disconnected setting is, as far as we understand, still open. In the discrete case, Palmer [35] claims that all examples of discrete groups which are known to be Hermitian actually have subexponential growth (even more, polynomial growth).

An affirmative answer to question 3.3.2 would make, as we pointed out, the class of groups with a quasi-symmetric group algebra richer than the union of the classes of Hermitian and subexponential growth groups.

On the other side, a negative answer to the above question would mean that any Hermitian totally disconnected group necessarily has subexponential growth and is therefore amenable, and thus would bring new evidence for the long standing conjecture that all Hermitian groups are amenable ([35]), which is known to be true in the connected case [35, Theorem 12.5.18 (e)]. In fact, a negative answer to 3.3.2 in the discrete case alone would, through the theory of extensions, imply that all Hermitian groups with an open connected component are amenable.

The fact that we do not know of any totally disconnected group with a quasi-symmetric group algebra which does not have subexponential growth is not a drawback in any way. In fact, the class of groups with subexponential growth is already very rich by itself and will be used to give meaningful examples in Hecke C^*-algebra theory and Hall's equivalence in the next section.

3.4 Hall's equivalence

Combining the results of Chapter 2 on the existence of $C^*(G,\Gamma)$ and the isomorphism $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$, together with the result of this chapter on the isomorphism $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ and Tzanev's theorem on the isomorphism $pC^*(\overline{G})p \cong C^*_r(G,\Gamma)$, we are able to establish that

$$C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C^*_r(G,\Gamma)\,,$$

for several classes of Hecke pairs, including all Hecke pairs (G,Γ) where G is a nilpotent group. As consequence, Kaliszewski, Landstad and Quigg's theorem (Theorem 1.2.14 in the present work) yields that Hall's equivalence is satisfied for all such classes of Hecke pairs.

Proposition 3.4.1. *If a group G satisfies one of the following generalized nilpotency properties:*

- *G is finite-by-nilpotent, or*
- *G is hypercentral, or*
- *all subgroups of G are subnormal,*

then for any Hecke subgroup $\Gamma \subseteq G$ we have that $C^*(G,\Gamma)$ exists and

$$C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C^*_r(G,\Gamma)\,.$$

In particular, Hall's equivalence holds with respect to any Hecke subgroup.

Proof: As discussed in subsections 2.4.8, 2.4.9 and 2.4.5, for every Hecke pair (G,Γ) where G satisfies one of the aforementioned properties we have that every double coset generates a finite co-hereditary set, and therefore the full Hecke C^*-algebra exists and we have $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$.

We now claim that if G has one of the three properties above, it must have subexponential growth. If G is finite-by-nilpotent, then by definition G is a nilpotent extension of a finite group, and since nilpotent groups have subexponential growth, then so does G. If G is hypercentral or all subgroups of G are subnormal, then it is known that G is locally nilpotent and therefore must have subexponential growth (see [22]). Consequently, the group algebra of G is quasi-symmetric and we have $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$.

The last isomorphism, $pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$, is obtained via Tzanev's theorem (Theorem 1.2.10 in the present work) as we describe now. Since any group G with one of the above properties has subexponential growth, it is therefore amenable. Amenability of G implies the amenability of (G,Γ), as was observed in [13, Exposé n° 1, §3], and hence Tzanev's theorem can be applied. □

If we restrict ourselves to finite subgroups $\Gamma \subseteq G$ we get a similar result for other classes of groups:

Proposition 3.4.2. *If a group G satisfies one of the following properties:*

- *G is an FC-group, or*
- *G is locally nilpotent, or*
- *G is locally finite,*

then for any finite subgroup $\Gamma \subseteq G$ we have that $C^*(G,\Gamma)$ exists and

$$C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p \cong C_r^*(G,\Gamma)\,.$$

In particular, Hall's equivalence holds with respect to any finite subgroup.

Proof: As discussed in subsections 2.4.10, 2.4.11 and 2.4.12, for every group G that satisfies one of the aforementioned properties and every finite subgroup $\Gamma \subseteq G$ we have that every double coset generates a finite cohereditary set, and therefore the full Hecke C^*-algebra exists and we have $C^*(G,\Gamma) \cong C^*(L^1(G,\Gamma))$

We have that if G has one of the three properties above, it must have subexponential growth (for FC- and locally nilpotent groups see [22], and for locally finite groups it is obvious). Consequently, the group algebra of G is quasi-symmetric and we have $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$.

The last isomorphism, $pC^*(\overline{G})p \cong C_r^*(G,\Gamma)$, is obtained via Tzanev's theorem (Theorem 1.2.10 in the present work) as we describe now. Since any group G with one of the above properties has subexponential growth, it is therefore amenable. Amenability of G then implies the amenability of (G,Γ), as was observed in [13, Exposé n° 1, §3], and hence Tzanev's theorem can be applied. □

Remark 3.4.3. The results above show that Hall's equivalence holds for any Hecke pair (G, Γ) where G satisfies a certain generalized nilpotency property. An analogous result for the class of solvable groups cannot hold. In [42, Example 3.4] Tzanev gave an example of a Hecke pair (G, Γ) where G is solvable but for which $C^*(G, \Gamma)$ does not exist, and consequently Hall's equivalence does not hold. The example consists of the infinite dihedral group $G := \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ together with $\Gamma := \mathbb{Z}/2\mathbb{Z}$.

3.5 A counter-example

In the previous sections we have established a sufficient condition for the isomorphism $C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p$ to hold, namely whenever $\overline{G}$ has a quasi-symmetric group algebra. A natural question to ask is the following: is it even possible that $C^*(L^1(G, \Gamma)) \ncong pC^*(\overline{G})p$? According to [24, page 677], Tzanev claims in private communication with Kaliszewski, Landstad and Quigg that the Hecke pair $(PSL_3(\mathbb{Q}_q), PSL_3(\mathbb{Z}_q))$ gives one such example, but no proof has been published and no other example is known, as far as we know. Here q denotes a prime number and $\mathbb{Q}_q$, $\mathbb{Z}_q$ denote respectively the field of q-adic numbers and the ring of q-adic integers.

In this section we will show that $C^*(L^1(G, \Gamma)) \ncong pC^*(\overline{G})p$ for the Hecke pair $(PSL_2(\mathbb{Q}_q), PSL_2(\mathbb{Z}_q))$, as it was asked and suggested in [24, Example 10.8]. Our approach is nevertheless different from the approach suggested in [24], since we make no use of the representation theory of $PSL_2(\mathbb{Q}_q)$.

Theorem 3.5.1. *Let q be a prime number and $\mathbb{Q}_q$ and $\mathbb{Z}_q$ denote respectively the field of q-adic numbers and the ring of q-adic integers. For the Hecke pair $(G, \Gamma) := (PSL_2(\mathbb{Q}_q), PSL_2(\mathbb{Z}_q))$ we have that $C^*(L^1(G, \Gamma)) \ncong pC^*(G)p$.*

Proof: For ease of reading and so that no confusion arises between the prime number q and the projection p, we will throughout this proof denote the projection p by P. Thus, our goal is to prove that $C^*(L^1(G, \Gamma)) \ncong PC^*(G)P$.

The pair $(PSL_2(\mathbb{Q}_q), PSL_2(\mathbb{Z}_q))$ coincides with its own Schlichting completion (see [24]) and is the reduction of the pair $(SL_2(\mathbb{Q}_q), SL_2(\mathbb{Z}_q))$. For ease of reading we will work with pair $(SL_2(\mathbb{Q}_q), SL_2(\mathbb{Z}_q))$ in this proof.

The structure of the Hecke algebra $\mathcal{H}(G, \Gamma)$ is well-known, and for convenience we will mostly refer to Hall [18, Section 2.1.2.1] whenever we need to. Letting

$$x_n := \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix} ,$$

it is known ([18, Prop. 2.9]) that every double coset $\Gamma s \Gamma$ can be uniquely represented as $\Gamma x_n \Gamma$ for some $n \in \mathbb{N}$.

For each $0 \leq k \leq q-1$ let us denote by $y_k \in G$ the matrix

$$y_k := \begin{pmatrix} q & k \\ 0 & q^{-1} \end{pmatrix} ,$$

and let us take $g \in L^1(G)P$ as the element $g := y_0 P + y_1 P + \cdots + y_{q-1}P$, and $f := P + g$. We then have

$$\begin{aligned} f^* f &= (P+g)^*(P+g) \;=\; P + g^*P + Pg + g^*g \\ &= P + \sum_{k=0}^{q-1} P y_k^{-1} P + \sum_{k=0}^{q-1} P y_k P + \sum_{i,j=0}^{q-1} P y_i^{-1} y_j P \\ &= (q+1)P + \sum_{k=0}^{q-1} P y_k^{-1} P + \sum_{k=0}^{q-1} P y_k P + \sum_{\substack{i,j=0 \\ i \neq j}}^{q-1} P y_i^{-1} y_j P \,. \end{aligned}$$

As it is know (see for example [18, Props. 2.10 and 2.12]), in $\mathcal{H}(G,\Gamma)$ the modular function is trivial and each double coset is self-adjoint. Hence we can write

$$f^* f \;=\; (q+1)P + 2\sum_{k=0}^{q-1} P y_k P + 2 \sum_{\substack{i,j=0 \\ i<j}}^{q-1} P y_i^{-1} y_j P \,.$$

We now notice that, from [18, Prop. 2.9], we have $\Gamma y_k \Gamma = \Gamma x_1 \Gamma$, and therefore $P y_k P = P x_1 P$. Moreover, for $0 \leq i < j \leq q-1$, we have that

$$y_i^{-1} y_j = \begin{pmatrix} 1 & (j-i)q^{-1} \\ 0 & 1 \end{pmatrix} ,$$

and again from [18, Prop. 2.9] we conclude that $P y_i^{-1} y_j P = P x_1 P$. Hence, we get

$$\begin{aligned} f^* f &= (q+1)\, P + 2q \, P x_1 P + 2 \frac{(q-1)q}{2} \, P x_1 P \\ &= (q+1)\, P + (q^2+q)\, P x_1 P \,. \end{aligned}$$

It is well known that $\mathcal{H}(G,\Gamma)$ is commutative (see for example [18, Section 2.2.3.2]) and all of its characters have been explicitly described. Following [24, Example 10.8] the characters of $\mathcal{H}(G,\Gamma)$ are precisely all the functions $\pi_z : \mathcal{H}(G,\Gamma) \to \mathbb{C}$ such that

$$\pi_z(P x_m P) = \frac{1-qz}{(q+1)(1-z)} \Big(\frac{z}{q}\Big)^m + \frac{q-z}{(q+1)(1-z)} \Big(\frac{1}{qz}\Big)^m ,$$

for a given complex number $z \in \mathbb{C}\backslash\{1\}$ (the expression for π_1 is different and the reader should check [24, Example 10.8] for the correct definition, but we will not need it here). Kaliszewski, Landstad and Quigg [24, Example 10.8] have also determined that the characters π_z which extend to *-representations of $L^1(G,\Gamma)$ are precisely those with $z \in [-q,-1/q] \cup [1/q,q]$.

We will now consider the *-representation π_{-q} of $L^1(G,\Gamma)$ and show that $\pi_{-q}(f^*f) < 0$. First we notice that

$$\begin{aligned}\pi_{-q}(Px_1P) &= \frac{1-q(-q)}{(q+1)(1-(-q))}\Big(\frac{-q}{q}\Big) + \frac{q-(-q)}{(q+1)(1-(-q))}\Big(\frac{1}{q(-q)}\Big)\\ &= -\frac{1+q^2}{(q+1)^2} - \frac{2}{(q+1)^2 q}\\ &= -\frac{q^3+q+2}{(q+1)^2 q}\,.\end{aligned}$$

Hence we get

$$\begin{aligned}\pi_{-q}(f^*f) &= \pi_{-q}\big((q+1)\,P + (q^2+q)\,Px_1P\big)\\ &= q+1-(q^2+q)\frac{q^3+q+2}{(q+1)^2 q}\\ &= q+1-\frac{q^3+q+2}{q+1}\,.\end{aligned}$$

To prove that $\pi_{-q}(f^*f) < 0$ is then equivalent to show that $(q+1)^2 < q^3+q+2$, or equivalently, $0 < q^3-q^2-q+1$, for any prime number q. This follows from an elementary calculus argument as follows: letting $F(x) = x^3-x^2-x+1$, we have that $F''(x) = 6x-2$ is always greater than 0 for $x \geq 2$ (the first prime number). Hence, $F'(x) = 3x^2-2x-1$ is growing for $x \geq 2$. Since $F'(2) > 0$, it follows that $F'(x)$ is always greater than 0 for $x \geq 2$. Thus, $F(x)$ is growing in this interval, and since $F(2) > 0$, it follows that $F(q) > 0$, for any prime q.

Since $\pi_{-q}(f^*f) < 0$ it then follows that not all representations of $L^1(G,\Gamma)$ are $\langle\rangle_R$-positive and consequently $C^*(L^1(G,\Gamma)) \ncong PC^*(G)P$. $\square$

As a particular consequence of the above theorem, it follows that $PSL_2(\mathbb{Q}_q)$ does not have a quasi-symmetric group algebra. Also, together with Hall's result [18, Proposition 2.21] and the fact that $PSL_2(\mathbb{Q}_q)$ is not amenable, we can say that for this Hecke pair $C^*(G,\Gamma)$ does not exist and $C^*(L^1(G,\Gamma)) \ncong pC^*(\overline{G})p \ncong C^*_r(G,\Gamma)$.

As we have seen in this chapter, the isomorphism $C^*(L^1(G,\Gamma)) \cong pC^*(\overline{G})p$ holds whenever G (G_r or $\overline{G}$) has subexponential growth. We would like know if the same is true or if one counter-example can be found for the class of amenable groups:

Open Question 3.5.2. If the pair (G, Γ) is amenable (equivalently, $\overline{G}$ is amenable), does it follow that $C^*(L^1(G, \Gamma)) \cong pC^*(\overline{G})p$?

Part II

Crossed products by Hecke pairs

Chapter 4

Preliminaries

In this chapter we set up the conventions, notation, and background results which will be used throughout this second part of the thesis. The topics covered include: essential *-algebras; *-algebraic multiplier algebras; Fell bundles over discrete groupoids. We indicate the references where the reader can find more details, but we also provide proofs for those results which we could not find in the literature.

4.1 Essential *-algebras

In this section we introduce the notion of an *essential* *-algebra. The class of essential *-algebras seems to be the appropriate class of *-algebras for which one can a define a multiplier algebra (as we shall see in Section 4.2).

We start with the definition of semiprimeness taken from Palmer [34, Definition 4.4.1].

Definition 4.1.1. An algebra A is said to be *semiprime* if $\{0\}$ is the only nilpotent ideal in A. Recall that an ideal $I \subseteq A$ is said to be *nilpotent* if there is some $n \in \mathbb{N}$ such that $I^n = \{0\}$.

There are several well-known equivalent definitions of a semiprime algebra. The following result, which is a particular instance of [34, Theorem 4.4.3], lists these equivalent properties:

Proposition 4.1.2. *Let A be an algebra. The following are equivalent:*

a) A is semiprime.

b) $aAa = \{0\}$ *implies* $a = 0$*, for all* $a \in A$*.*

c) If I *is an ideal of* A*, then* $I^2 = \{0\}$ *implies* $I = \{0\}$*.*

d) If J *is a one-sided ideal of* A *satisfying* $J^n = \{0\}$ *for some* $n \in \mathbb{N}$*, then* $J = \{0\}$*.*

e) $\{0\}$ *is the intersection of some set of prime ideals.*

We will say that a *-algebra is *semiprime* if its underlying algebra is semiprime. It is not difficult to see, from property *b*), that a *-algebra that admits a faithful *-representation on a Hilbert space is always semiprime. For more information on how semiprimeness relates with other properties of *-algebras the reader should consult [35, Theorem 9.7.21].

We now give a definition of an *essential ideal*.

Definition 4.1.3. Let A be an algebra. An ideal $I \subseteq A$ is said to be *essential* if $aI \neq \{0\}$ for all $a \in A \setminus \{0\}$.

The reader may find that this definition is not the usual one, where an I is said to be essential if it has nonzero intersection with every other nonzero ideal. In fact our definition is stronger, but coincides with the usual one for semiprime *-algebras (in particular, for C^*-algebras), as it is shown below.

Proposition 4.1.4. *Let* A *be an algebra and* $I \subseteq A$ *a nonzero ideal. We have*

i) If I *is essential, then* I *has a nonzero intersection with every other nonzero ideal of* A*.*

ii) The converse of i) *is true in case* A *is semiprime.*

Proof: *i*) Let I be an essential ideal of A. Let $J \subseteq A$ be a nonzero ideal and $a \in J \setminus \{0\}$. Since a is nonzero, then $aI \neq \{0\}$. Hence, $J \cdot I \neq \{0\}$, and since $J \cdot I \subseteq J \cap I$, we have $J \cap I \neq \{0\}$.

ii) Suppose A is semiprime. Suppose also that I is not essential. Thus, there is $a \in A \setminus \{0\}$ such that $aI = \{0\}$. Let $J_a \subseteq A$ be the ideal generated by a. We have $J_a \cdot I = \{0\}$. Since $(J_a \cap I)^2 \subseteq J_a \cdot I$ we have $(J_a \cap I)^2 = \{0\}$, and consequently since A is semiprime, $J_a \cap I = \{0\}$. Hence, I has zero intersection with a nonzero ideal. □

For C^*-algebras the focus is mostly on closed ideals. In this setting we still see that our definition is equivalent to the usual one ([37, Definition 2.35]):

Proposition 4.1.5. *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal. The following are equivalent:*

i) I is essential.

ii) I has nonzero intersection with every other nonzero ideal of A.

iii) I has nonzero intersection with every other nonzero closed ideal of A.

Proof: *i)* $\Longleftrightarrow$ *ii)* This was established in Proposition 4.1.4, since C^*-algebras are automatically semiprime.

ii) $\Longrightarrow$ *iii)* This is obvious.

ii) $\Longleftarrow$ *iii)* Let J be a nonzero ideal of A and $\overline{J}$ its closure. From *iii)* we have $I \cap \overline{J} \neq \{0\}$. Since I and $\overline{J}$ are both closed, and A is a C^*-algebra, we have $I \cdot \overline{J} = I \cap \overline{J}$. Now, it is clear that $I \cdot J = \{0\}$ if and only if $I \cdot \overline{J} = \{0\}$. Hence, we necessarily have $I \cdot J \neq \{0\}$, which implies $I \cap J \neq \{0\}$. □

Definition 4.1.6. A *-algebra A is said to be *essential* if A is an essential ideal of itself, i.e. if $aA \neq \{0\}$ for all $a \in A \setminus \{0\}$.

Any unital *-algebra is obviously essential. Also, it is easy to see that a semiprime *-algebra is essential, by property *b)* of Proposition 4.1.4. The converse is false, so that essential *-algebras form a more general class than that of semiprime algebras:

Example 4.1.7. Let $\mathbb{C}[X]$ be the polynomial algebra in one variable X. For any $n \geq 2$ the algebra $\mathbb{C}[X]/\langle X^n \rangle$ is essential, because it is unital, but it is not semiprime because $[X^{n-1}]\Big(\mathbb{C}[X]/\langle X^n \rangle\Big)[X^{n-1}] = \{0\}$.

The following diagram summarizes how different classes of *-algebras are related:

$$\begin{gathered}\{C^*\text{-algebras}\}\\ \Downarrow\\ \{{}^*\text{-algebras that have a faithful representation}\}\\ \Downarrow\\ \{\text{semiprime }{}^*\text{-algebras}\}\\ \Downarrow\\ \{\text{essential }{}^*\text{-algebras}\}\end{gathered}$$

4.2 *-Algebraic multiplier algebras

Every C^*-algebra can be embedded in a unital C^*-algebra in a "maximal" way. These maximal unitizations of C^*-algebras enjoy a number of useful properties and some concrete realizations of these algebras are commonly referred to as multiplier algebras. The reader is referred to [37] for an account.

The definition of a multiplier algebra is quite standard in C^*-algebra theory, but this notion is in fact more general and applicable for more general types of rings and algebras. For example, in [1, Section 1.1] it is explained how multiplier algebras can be defined for semiprime algebras.

In this section we are going to generalize this notion to the context of essential *-algebras and derive their basic properties. We believe that essential *-algebras are the appropriate class of *-algebras for which one can define multiplier algebras, since the property $aA = \{0\} \Rightarrow a = 0$, which characterizes an essential *-algebra, is constantly used in proofs.

Multiplier algebras are many times defined via the so-called double centralizers (see for example [1]), but since we are only interested in algebras with an involution a slightly simpler and more convenient approach can be given, analogue to the Hilbert C^*-module approach to C^*-multiplier algebras (presented in [37]). This is the approach we follow.

Definition 4.2.1. Let $\mathcal{C}$ be a subclass of *-algebras. A *-algebra $A \in \mathcal{C}$ is said to have a *maximal unitization in* $\mathcal{C}$ if there exists a unital *-algebra $B \in \mathcal{C}$ (called the *maximal unitization* of A) and a *-embedding $i : A \hookrightarrow B$ for which $i(A)$ is an essential ideal of B and such that for every other *-embedding j of A as an essential ideal of a unital *-algebra $C \in \mathcal{C}$, there is a unique *-homomorphism $\phi : C \to B$ such that

$$\begin{tikzcd} & B \\ A \arrow[ur, "i"] \arrow[r, "j"'] & C \arrow[u, dotted, "\phi"'] \end{tikzcd}$$

commutes.

Remark 4.2.2. The universal property above implies that the maximal unitization is unique up to *-isomorphism. In language of category theory, a maximal unitization of $A \in \mathcal{C}$ is the same as a terminal object in the category of essential unitizations of A (in $\mathcal{C}$). In the same way, a minimal unitization is an initial object in this category.

Lemma 4.2.3. *In the above diagram the *-homomorphism ϕ is always injective (even if C was not unital).*

Proof: We have that $j(A) \cap \operatorname{Ker} \phi = \{0\}$, because if $j(a) \in j(A) \cap \operatorname{Ker} \phi$, then $0 = \phi(j(a)) = i(a)$ and hence $a = 0$ and therefore $j(a) = 0$. Hence, since $j(A)$ is an essential ideal of C, it follows from Proposition 4.1.4 *i)* that $\operatorname{Ker} \phi = \{0\}$. $\square$

For C^*-algebras, one might expect to replace "ideal" by "closed ideal", in Definition 4.2.1. This condition, however, follows automatically since $i(A)$ and $j(A)$ are automatically closed. Hence, this definition encompasses the usual definition of a maximal unitization for a C^*-algebra.

Definition 4.2.4. Let A be a *-algebra. By a *right A-module* we mean a vector space X together with a mapping $X \times A \to X$ satisfying the usual consistency conditions. An *A-linear mapping* $T : X \to Y$ between A-modules is a linear mapping between the underlying vector spaces such that $T(xa) = T(x)a$, for all $x \in X$ and $a \in A$. We will often use the notation Tx, instead of $T(x)$.

Every *-algebra A is canonically a right A-module, with the action of right multiplication. This is the example we will use thoroughly in what follows.

Let $\langle \cdot, \cdot \rangle_A : A \times A \to A$ be the function

$$\langle a, b \rangle_A := a^* b \,.$$

The function $\langle \cdot, \cdot \rangle_A$ is an A-linear form, in the sense that the following properties are satisfied:

a) $\langle a \,,\, \lambda_1 b_1 + \lambda_2 b_2 \rangle_A = \lambda_1 \langle a, b_1 \rangle_A + \lambda_2 \langle a, b_2 \rangle_A \,,$

b) $\langle \lambda_1 a_1 + \lambda_2 a_2 \,,\, b \rangle_A = \overline{\lambda_1} \langle a_1, b \rangle_A + \overline{\lambda_2} \langle a_2, b \rangle_A \,,$

c) $\langle a, bc \rangle_A = \langle a, b \rangle_A \, c \,,$

d) $\langle ac, b \rangle_A = c^* \langle a, b \rangle_A \,,$

e) $\langle a, b \rangle_A^* = \langle b, a \rangle_A \,,$

for all $a, a_1, a_2, b, b_1, b_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

If the *-algebra A is essential we also have:

f) If $\langle a \,,\, b \rangle_A = 0$ for all $b \in A$, then $a = 0$.

Definition 4.2.5. Let A be a *-algebra. A function $T : A \to A$ is called *adjointable* if there is a function $T^* : A \to A$ such that

$$\langle T(a), b \rangle_A = \langle a, T^*(b) \rangle_A \,,$$

for all $a, b \in A$.

Proposition 4.2.6. *If A is an essential *-algebra, then every adjointable map $T : A \to A$ is A-linear. Moreover, the adjoint T^* is unique and adjointable with $T^{**} = T$.*

Proof: Let T be an adjointable map in A and $x_1, x_2, y \in A$. We have

$$\begin{aligned} \langle T(\lambda_1 x_1 + \lambda_2 x_2) \,,\, y \rangle_A &= \langle \lambda_1 x_1 + \lambda_2 x_2 \,,\, T^*(y) \rangle_A \\ &= \overline{\lambda_1} \, \langle x_1 \,,\, T^*(y) \rangle_A + \overline{\lambda_2} \, \langle x_2 \,,\, T^*(y) \rangle_A \\ &= \overline{\lambda_1} \, \langle T(x_1) \,,\, y \rangle_A + \overline{\lambda_2} \, \langle T(x_2) \,,\, y \rangle_A \\ &= \langle \lambda_1 T(x_1) + \lambda_2 T(x_2) \,,\, y \rangle_A \,. \end{aligned}$$

Hence, we have $\langle T(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T(x_1) + \lambda_2 T(x_2)\, ,\, y\rangle_A = 0$. We can then conclude from f) that

$$T(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T(x_1) + \lambda_2 T(x_2) = 0\,,$$

i.e. T is a linear map.

Let us now check that T is A-linear. For any $x, y, a \in A$ we have

$$\begin{aligned} \langle T(xa)\, ,\, y\rangle_A &= \langle xa\, ,\, T^*(y)\rangle_A = a^*\langle x\, ,\, T^*(y)\rangle_A \\ &= a^*\langle T(x)\, ,\, y\rangle_A = \langle T(x)a\, ,\, y\rangle_A\,. \end{aligned}$$

Hence, we have $\langle T(xa) - T(x)a\, ,\, y\rangle_A = 0$. We can then conclude from f) that $T(xa) - T(x)a = 0$, i.e. T is A-linear.

Let us now prove the uniqueness of the adjoint T^*. Suppose there was a function $S : A \to A$ such that

$$\langle x\, ,\, T^*(y)\rangle_A = \langle x\, ,\, S(y)\rangle_A\,.$$

for all $x, y \in A$. Then, $\langle T^*(y) - S(y)\, ,\, x\rangle_A = 0$. We can then conclude from f) that $T^*(y) - S(y) = 0$, i.e. $T^* = S$.

It remains to prove that T^* is adjointable with $T^{**} = T$. This follows easily from the equality

$$\langle T^* x\, ,\, y\rangle_A = \langle y\, ,\, T^* x\rangle_A^* = \langle Ty\, ,\, x\rangle_A^* = \langle x, Ty\rangle_A\,.$$

$\square$

Definition 4.2.7. Let A be an essential *-algebra. The set of all adjointable maps on A is called the *multiplier algebra* of A and is denoted by $M(A)$.

The multiplier algebra is in fact a *-algebra, and the proof of this fact is standard.

Proposition 4.2.8. *Let A be an essential *-algebra. The multiplier algebra of A is an essential *-algebra with the sum and multiplication given by pointwise sum and composition (respectively), and the involution given by the adjoint.*

Proposition 4.2.9. *Let A be an essential *-algebra. There is a *-embedding $L : A \to M(A)$ of A as an essential ideal of $M(A)$, given by*

$$a \mapsto L_a$$

where $L_a : A \to A$ is the left multiplication by a, i.e. $L_a(b) := ab$.

Proof: It is easy to see that, for every $a \in A$, L_a is adjointable with adjoint L_{a^*}, thus the mapping L is well-defined. Also clear is the fact that L is a *-homomorphism. Let us prove that it is injective: suppose $L_a = 0$ for some $a \in A$. Then, for all $b \in A$ we have $ab = L_a b = 0$ and since A is essential this implies $a = 0$. Thus, L is injective.

It remains to prove that $L(A)$ is an essential ideal of $M(A)$. Let us begin by proving that it is an ideal. Let $T \in M(A)$. For every $a, b \in A$ we have

$$TL_a(b) = T(ab) = T(a)b = L_{Ta}(b)\,,$$

and also

$$\begin{aligned} L_aT(b) &= aT(b) = \langle a^*, T(b)\rangle \\ &= \langle T^*(a^*), b\rangle = (T^*(a^*))^*b \\ &= L_{(T^*a^*)^*}(b)\,. \end{aligned}$$

Hence we have

$$TL_a = L_{Ta} \qquad \text{and} \qquad L_aT = L_{(T^*a^*)^*}\,, \tag{4.1}$$

from which it follows easily that $L(A)$ is an ideal of $M(A)$.

Let us now prove that this ideal is essential. Let $T \in M(A)$ be such that $TL(A) = \{0\}$. Then, in particular, $TL_a = 0$ for all $a \in A$, but as we have seen before $TL_a = L_{Ta}$, and since L is injective we must have $Ta = 0$ for all $a \in A$, i.e $T = 0$. $\square$

Remark 4.2.10. According to Proposition 4.2.9, an essential *-algebra A is canonically embedded in its multiplier algebra $M(A)$. We will often make no distinction of notation between A and its embedded image in $M(A)$, i.e. we will often just write a to denote an element of A and to denote the element $L(a)$ of $M(A)$. No confusion will arise from this because the left equality in (4.1) simply means, in this notation, that $T \cdot a = T(a)$.

Theorem 4.2.11. *Let A be an essential *-algebra and $L : A \to M(A)$ the canonical *-embedding of A in $M(A)$. If $j : A \to C$ is a *-embedding of A as an ideal of a *-algebra C, then there exists a unique *-homomorphism $\phi : C \to M(A)$ such that the following diagram commutes*

$$\begin{tikzcd} & M(A) \\ A \arrow[ur, "L"] \arrow[r, "j"'] & C \arrow[u, dotted, "\phi"'] \end{tikzcd}$$

Moreover, if $j(A)$ is essential then ϕ is injective.

Proof: For simplicity of notation let us assume, without any loss of generality, that A itself is an ideal of a *-algebra C, so that we avoid any reference to j (or its inverse). Let $\phi : C \to M(A)$ be the function defined by

$$\begin{aligned} \phi(c) &: A \to A \\ \phi(c)a &:= ca\,, \end{aligned}$$

for every $c \in C$. It is a straightforward computation to check that $\phi(c) \in M(A)$ and that ϕ itself is a *-homomorphism. It is also easy to see that $\phi(a) = L_a$, for every $a \in A$. Hence, $\phi \circ j = L$. Let us now prove the uniqueness of ϕ relatively to this property. Suppose $\widetilde{\phi} : C \to M(A)$ is another *-homomorphism such that $\widetilde{\phi} \circ j = L$. Then, for all $c \in C$ and $a \in A$ we have

$$\begin{aligned} \big(\widetilde{\phi}(c) - \phi(c)\big)L_a &= \widetilde{\phi}(c)L_a - \phi(c)L_a \\ &= \widetilde{\phi}(c)\widetilde{\phi}(a) - \phi(c)\phi(a) \\ &= \widetilde{\phi}(ca) - \phi(ca) \\ &= L_{ca} - L_{ca} \\ &= 0\,. \end{aligned}$$

Since $L(A)$ is an essential ideal of $M(A)$ it follows that $\widetilde{\phi}(c) = \phi(c)$ for all $c \in C$, i.e. $\widetilde{\phi} = \phi$.

The last claim of the theorem, concerning injectivity of ϕ, was proven in Lemma 4.2.3. □

Corollary 4.2.12. *The multiplier algebra $M(A)$ is a maximal unitization of A in the class of: essential *-algebras, semiprime *-algebras and C^*-algebras.*

Proof: By Theorem 4.2.11 we only need to check that if A is an essential *-algebra (respectively, semiprime *-algebra or C^*-algebra), then the multiplier algebra has the same property.

Suppose A is an essential *-algebra. Let $T \in M(A)$ be such that $TM(A) = \{0\}$. Then, by the embedding of A in $M(A)$ we have $Ta = 0$ for all $a \in A$, i.e. $T = 0$. Hence, $M(A)$ is also an essential *-algebra.

Suppose A is a semiprime *-algebra. Let $T \in M(A)$ be such that $TM(A)T = \{0\}$. Then, we also have that $TL_a M(A) TL_a = \{0\}$ for any $a \in A$, and therefore $L_{T(a)} M(A) L_{T(A)} = \{0\}$. Thus, in particular, $L_{T(a)} L(A) L_{T(a)} = \{0\}$, and since L is injective this means that $T(a)AT(a) = \{0\}$. Since A is semiprime we conclude that $T(a) = 0$, and therefore $T = 0$. Hence, $M(A)$ is semiprime.

It is well-known that $M(A)$ is a C^*-algebra when A is a C^*-algebra. $\square$

An important feature of C^*-multiplier algebras is that a nondegenerate *-representation of A extends uniquely to $M(A)$. This result does not hold in general for essential *-algebras. Nevertheless we can still extend a nondegenerate *-representation of A to a unique pre-*-representation of $M(A)$:

Theorem 4.2.13. *Let A be an essential *-algebra, $\pi : A \to B(\mathscr{H})$ a nondegenerate *-representation of A on a Hilbert space $\mathscr{H}$ and $\mathscr{V} \subseteq \mathscr{H}$ the dense subspace*

$$\mathscr{V} := \pi(A)\mathscr{H} = \operatorname{span}\{\pi(a)\xi : a \in \mathcal{A}, \xi \in \mathscr{H}\}.$$

Then there is a unique pre-*-representation

$$\widetilde{\pi} : M(A) \to L(\mathscr{V})$$

such that $\widetilde{\pi}(a) = \pi(a)|_{\mathscr{V}}$ for every $a \in A$.

Proof: We define the pre-*-representation $\widetilde{\pi} : M(A) \to L(\mathscr{V})$ by

$$\widetilde{\pi}(T)\,\Big[\sum_{i=1}^{n} \pi(a_i)\xi_i\Big] := \sum_{i=1}^{n} \pi(Ta_i)\xi_i\,,$$

for $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$ and $\xi_1, \dots, \xi_n \in \mathscr{H}$. Let us first check that $\widetilde{\pi}$ is well-defined. Suppose $\sum_{i=1}^{n} \pi(a_i)\xi_i = \sum_{j=1}^{m} \pi(b_j)\eta_j$. Then, for every $z \in A$

we have

$$\begin{aligned}
\pi(z)\Big(\sum_{i=1}^{n}\pi(Ta_i)\xi_i - \sum_{j=1}^{m}\pi(Tb_i)\eta_j\Big) &= \sum_{i=1}^{n}\pi(zTa_i)\xi_i - \sum_{j=1}^{m}\pi(zTb_i)\eta_j \\
&= \pi(zT)\Big(\sum_{i=1}^{n}\pi(a_i)\xi_i - \sum_{j=1}^{m}\pi(b_i)\eta_j\Big) \\
&= 0\,.
\end{aligned}$$

Since the *-representation π is nondegenerate we necessarily have

$$\sum_{i=1}^{n}\pi(Ta_i)\xi_i - \sum_{j=1}^{m}\pi(Tb_i)\eta_j = 0\,,$$

which means that $\widetilde{\pi}(T)$ is well-defined.

Let us now check that $\widetilde{\pi}(T) \in L(\mathscr{V})$, i.e. that $\widetilde{\pi}(T)$ is indeed an adjointable operator in $\mathscr{V}$. We will in fact prove that $\widetilde{\pi}(T)^* = \widetilde{\pi}(T^*)$, which follows from the following equality

$$\begin{aligned}
\Big\langle \widetilde{\pi}(T)\sum_{i=1}^{n}\pi(a_i)\xi_i\,,\,\sum_{j=1}^{m}\pi(b_i)\eta_j\Big\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle \pi(Ta_i)\xi_i\,,\,\pi(b_j)\eta_j\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle \xi_i\,,\,\pi(a_i^*T^*)\pi(b_j)\eta_j\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle \xi_i\,,\,\pi(a_i^*T^*b_j)\eta_j\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\langle \pi(a_i)\xi_i\,,\,\pi(T^*b_j)\eta_j\rangle \\
&= \Big\langle \sum_{i=1}^{n}\pi(a_i)\xi_i\,,\,\widetilde{\pi}(T^*)\sum_{j=1}^{m}\pi(b_i)\eta_j\Big\rangle\,.
\end{aligned}$$

It is straightforward to see that $\widetilde{\pi}$ is linear, multiplicative and, as we have seen, $\widetilde{\pi}(T^*) = \widetilde{\pi}(T)^*$, hence $\widetilde{\pi}$ is a pre-*-representation of $M(A)$ on $\mathscr{V}$.

It is also clear that, for any $a \in A$, $\widetilde{\pi}(a)$ is just $\pi(a)$ restricted to $\mathscr{V}$, because of the equality

$$\widetilde{\pi}(a)\sum_{i=1}^{n}\pi(a_i)\xi_i = \sum_{i=1}^{n}\pi(aa_i)\xi_i = \pi(a)\sum_{i=1}^{n}\pi(a_i)\xi_i\,.$$

Let us now prove the uniqueness of $\widetilde{\pi}$. Suppose $\phi : M(A) \to L(\mathscr{V})$ is a pre-*-representation such that $\phi(a) = \pi(a)|_{\mathscr{V}}$. Then, for every $z \in A$ and $v \in \mathscr{V}$ we have

$$\begin{aligned}
\pi(z)(\phi(T)v - \widetilde{\pi}(T)v) &= \pi(z)\phi(T)v - \pi(z)\widetilde{\pi}(T)v \\
&= \phi(z)\phi(T)v - \widetilde{\pi}(z)\widetilde{\pi}(T)v \\
&= \phi(zT)v - \widetilde{\pi}(zT)v \\
&= \pi(zT)v - \pi(zT)v \\
&= 0\,.
\end{aligned}$$

Since the *-representation π is nondegenerate, we necessarily have

$$\phi(T)v - \widetilde{\pi}(T)v = 0\,,$$

which means that $\phi(T) = \widetilde{\pi}(T)$, i.e. $\phi = \widetilde{\pi}$. □

Remark 4.2.14. Theorem 4.2.13 can be interpreted in the following way: every nondegenerate *-representation $\pi : A \to B(\mathscr{H})$ can be extended to $M(A)$ by possibly unbounded operators, defined on the dense subspace $\pi(A)\mathscr{H}$.

Definition 4.2.15. Let A be an essential *-algebra. We will denote by $M_B(A)$ the subset of $M(A)$ consisting of all the elements $T \in M(A)$ such that $\widetilde{\pi}(T) \in B(\mathscr{V})$ for all nondegenerate *-representations $\pi : A \to B(\mathscr{H})$, where $\mathscr{V} := \pi(A)\mathscr{H}$ and $\widetilde{\pi}$ is the unique pre-*-representation extending π as in Proposition 4.2.13.

As stated in the next result, $M_B(A)$ is a *-subalgebra of $M(A)$. The advantage on working with $M_B(A)$ over $M(A)$ is that nondegenerate *-representations of A always extend to *-representations of $M_B(A)$.

Proposition 4.2.16. *Let A be an essential *-algebra. The set $M_B(A)$ is a *-subalgebra of $M(A)$ containing A. Moreover, if $\pi : A \to B(\mathscr{H})$ is a nondegenerate *-representation of A, then there exists a unique *-representation of $M_B(A)$ on $\mathscr{H}$ that extends π.*

Proof: Let $T, S \in M_B(A)$. Let $\pi : A \to B(\mathscr{H})$ be any nondegenerate *-representation of A and $\widetilde{\pi}$ its extension to $L(\mathscr{V})$, in the sense of Theorem 4.2.13, where $\mathscr{V} := \pi(A)\mathscr{H}$. By definition, $\widetilde{\pi}(T), \widetilde{\pi}(S) \in B(\mathscr{V})$, and therefore

$\widetilde{\pi}(T+S), \widetilde{\pi}(TS), \widetilde{\pi}(T^*) \in B(\mathscr{V})$, since $B(\mathscr{V})$ is a *-algebra. Hence, $M_B(A)$ is a *-subalgebra of $M(A)$. Moreover, $A \subseteq M_B(A)$ since $\widetilde{\pi}(a) = \pi(a)|_{\mathscr{V}} \in B(\mathscr{V})$.

Let us now prove the last claim of this proposition. Let $\pi : A \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\pi} : M(A) \to L(\mathscr{V})$ its extension as in Theorem 4.2.13. Then we obtain by restriction a pre-*-representation $\widetilde{\pi} : M_B(A) \to L(\mathscr{V})$. By definition of $M_B(A)$ we actually have $\widetilde{\pi}(M_B(A)) \subseteq B(\mathscr{V})$. Hence $\widetilde{\pi}$ gives rise to a *-representation $\widetilde{\pi} : M_B(A) \to B(\mathscr{H})$, since $\mathscr{V}$ is dense in $\mathscr{H}$.

Let us now prove the uniqueness claim. Suppose φ is another representation of $M_B(A)$ that extends π. For $T \in M_B(A)$, $a \in A$ and $\xi \in \mathscr{H}$ we have

$$\begin{aligned} \varphi(T)\,\pi(a)\xi &= \varphi(T)\,\varphi(a)\xi = \varphi(Ta)\xi \\ &= \pi(Ta)\xi = \widetilde{\pi}(T)\,\pi(a)\xi\,. \end{aligned}$$

By linearity and density it follows that $\varphi(T) = \widetilde{\pi}(T)$, i.e. $\varphi = \widetilde{\pi}$. □

The above result is a generalization of the well-known result for C^*-algebras which states that any nondegenerate *-representation can be extended to the multiplier algebra (see for example [37, Corollary 2.51]), because $M(A) = M_B(A)$ for any C^*-algebra A.

4.3 Fell bundles over discrete groupoids

Let X be a discrete groupoid. We will denote by X^0 the unit space of X and by $\mathbf{s}$ and $\mathbf{r}$ the source and range functions $X \to X^0$, respectively.

Definition 4.3.1. Let A be a C^*-algebra and X a discrete groupoid. A *Fell bundle over* X is a collection $\mathcal{A} := \{\mathcal{A}_x\}_{x\in X}$ of closed subspaces of A indexed by X satisfying the following properties:

i) $\mathcal{A}_x\mathcal{A}_y \subseteq \mathcal{A}_{xy}$, i.e. if $x, y \in X$ are composable then for any $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$ we have $ab \in \mathcal{A}_{xy}$,

ii) $\mathcal{A}_x^* = \mathcal{A}_{x^{-1}}$, i.e. for any $a \in \mathcal{A}_x$ we have $a^* \in \mathcal{A}_{x^{-1}}$.

iii) $\mathcal{A}_x \neq \{0\}$ for all $x \in X^0$.

The subspace $\mathcal{A}_x$ is called the *fiber over* $x \in X$.

Remark 4.3.2. This is not the usual definition of a Fell bundle. The usual definition, found in [27], does not start with a fixed C^*-algebra A. Nevertheless, this definition is closely related with the standard one. On one hand, given a Fell bundle $\mathcal{A}$ over X in our sense one can naturally form a Fell bundle $\widetilde{\mathcal{A}}$ over X in the sense of [27], consisting of the disjoint union of all the fibers $\mathcal{A}_x$. The bundles $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ have isomorphic fibers and isomorphic *-algebras of finitely supported sections. On the other hand, given a Fell bundle $\mathcal{B}$ in the sense of [27], one can canonically form a Fell bundle in our sense, by taking A as the full cross sectional algebra $A := C^*(\mathcal{B})$ and $\mathcal{A}$ as the collection $\{\mathcal{B}_x\}_{x\in X}$ (seen as closed subspaces of $C^*(\mathcal{B})$, which is possible since X is discrete). The bundles $\mathcal{B}$ and $\mathcal{A}$ have isomorphic fibers and isomorphic algebras of sections.

Requirement *iii*), that all fibers over units are non-zero, is just a technical assumption, which is weaker than the notion of *nondegeneracy* for Fell bundles in the sense of [27, 2.2].

It is also important to remark that the above is not the definition of a groupoid grading of a C^*-algebra, because the subspaces $\{\mathcal{A}_x\}_{x\in X}$ are not assumed to be linearly independent. In fact, it will be the usual case in this work that some of the fibers are repeated, i.e. $\mathcal{A}_x = \mathcal{A}_y$ for some $x \neq y$. An easy example of this is the following: we consider a discrete groupoid X, a C^*-algebra A and each fiber $\mathcal{A}_x$ as A itself. In this way $\mathcal{A} := \{\mathcal{A}_x\}_{x\in X}$ is clearly a Fell bundle over X.

Remark 4.3.3. Even though in our definition of a Fell bundle we start with a fixed C^*-algebra A, it will not be very important in general to know what this C^*-algebra is. So, many times we will simply say "let $\mathcal{A}$ be a Fell bundle over X", without explicitly referring to the C^*-algebra A. It is thus implicitly assumed that A is defined and gives rise to the referred bundle $\mathcal{A}$.

Given a Fell bundle $\mathcal{A}$ over a discrete groupoid X we will denote by $C_c(\mathcal{A})$ its corresponding *-algebra of finitely supported sections. The following notation will be used throughout the rest of this work: for $x \in X$ and $a \in \mathcal{A}_x$ the symbol a_x will always denote the element of $C_c(\mathcal{A})$ such that

$$a_x(y) := \begin{cases} a\,, & \text{if } y = x \\ 0, & \text{otherwise}\,. \end{cases} \tag{4.2}$$

According to the notation above we can then write any $f \in C_c(\mathcal{A})$ uniquely as a sum of the form

$$f = \sum_{x \in X} (f(x))_x \,. \tag{4.3}$$

We recall that the operations of multiplication and involution in $C_c(\mathcal{A})$ are determined by

$$a_x \cdot b_y = \begin{cases} (ab)_{xy}\,, & \text{if } \mathbf{s}(x) = \mathbf{r}(y) \\ 0, & \text{otherwise}\,, \end{cases}$$
$$(a_x)^* = (a^*)_{x^{-1}}\,,$$

where $x, y \in X$ and $a \in \mathcal{A}_x$, $b \in \mathcal{A}_y$.

Given a Fell bundle $\mathcal{A}$ over a groupoid X we will denote by $\mathcal{A}^0$ the restricted bundle $\mathcal{A}|_{X^0}$ over the unit space X^0. Since the fibers of $\mathcal{A}$ over X^0 are C^*-algebras, $\mathcal{A}^0$ is a C^*-bundle over X^0.

It is known that the algebra $C_c(\mathcal{A})$ has an enveloping C^*-algebra (see for example [10, Proposition 2.1]). We now show a slightly stronger fact, that $C_c(\mathcal{A})$ is a BG^*-algebra, which will be useful later on.

Proposition 4.3.4. *Let X be a discrete groupoid and $\mathcal{A}$ a Fell bundle over X. The algebra $C_c(\mathcal{A})$ is a BG^*-algebra.*

Proof: Let π be a pre-*-representation of $C_c(\mathcal{A})$ on a inner product space $\mathscr{V}$. The algebra $C_c(\mathcal{A})$ is spanned by elements of the form a_x where $x \in X$ and $a \in \mathcal{A}_x$, so it suffices to show that $\pi(a_x) \in B(\mathscr{V})$. We have that

$$\|\pi(a_x)^*\pi(a_x)\| \;=\; \|\pi((a_x)^*a_x)\| \;=\; \|\pi((a^*a)_{\mathbf{s}(x)})\|\,.$$

Since $\mathcal{A}_{\mathbf{s}(x)}$ is a C^*-algebra, and C^*-algebras are BG^*-algebras, it follows that $\|\pi((a^*a)_{\mathbf{s}(x)})\| < \infty$, and therefore $\pi(a_x)^*\pi(a_x)$ is a bounded operator.

Let $\xi \in \mathscr{V}$ be a vector of norm $\|\xi\| \leq 1$. We have

$$\begin{aligned} \|\pi(a_x)\xi\|^2 \;&=\; \langle \pi(a_x)^*\pi(a_x)\xi\,,\,\xi\rangle \;\leq\; \|\pi(a_x)^*\pi(a_x)\xi\|\|\xi\| \\ &\leq\; \|\pi(a_x)^*\pi(a_x)\xi\| \;\leq\; \|\pi(a_x)^*\pi(a_x)\|\,. \end{aligned}$$

We conclude that $\pi(a_x) \in B(\mathscr{V})$, i.e. $C_c(\mathcal{A})$ is a BG^*-algebra. $\square$

We recall that the *full cross sectional algebra* of $\mathcal{A}$, denoted $C^*(\mathcal{A})$, is defined as the enveloping C^*-algebra of $C_c(\mathcal{A})$. If the groupoid X is just a set, in which case $\mathcal{A}$ is a C^*-bundle, we will use the notation $C_0(\mathcal{A})$ instead of $C^*(\mathcal{A})$.

We now recall, from [27], how the *reduced cross sectional algebra* $C_r^*(\mathcal{A})$ of a Fell bundle $\mathcal{A}$ over a (discrete) groupoid X is defined. We see $C_c(\mathcal{A})$ as a pre-Hilbert $C_0(\mathcal{A}^0)$-module, where the inner product is defined by

$$\langle f_1, f_2\rangle_{C_c(\mathcal{A}^0)} := (f_1^* \cdot f_2)|_{X^0}\,, \qquad\qquad f_1, f_2 \in C_c(\mathcal{A})\,.$$

Its completion is a full Hilbert $C_0(\mathcal{A}^0)$-module, which we denote by $L^2(\mathcal{A})$. Now, the algebra $C_c(\mathcal{A})$ acts on itself by left multiplication, and moreover this action is continuous with respect to the norm induced by the inner product above, hence we get an injective *-homomorphism

$$C_c(\mathcal{A}) \to \mathcal{L}(L^2(\mathcal{A}))\,. \tag{4.4}$$

The reduced cross sectional algebra $C_r^*(\mathcal{A})$ is defined as the completion of $C_c(\mathcal{A})$ with respect to the operator norm in $\mathcal{L}(L^2(\mathcal{A}))$, and in this way we get a right-Hilbert bimodule ${}_{C_r^*(\mathcal{A})}L^2(\mathcal{A})_{C_0(\mathcal{A}^0)}$.

Since $C_r^*(\mathcal{A})$ is a completion of $C_c(\mathcal{A})$ we immediately get a canonical map $\Lambda : C^*(\mathcal{A}) \to C_r^*(\mathcal{A})$. Also, the *-homomorphism above in (4.4) always completes to a *-homomorphism $C^*(\mathcal{A}) \to \mathcal{L}(L^2(\mathcal{A}))$, and therefore gives rise to a right-Hilbert bimodule ${}_{C^*(\mathcal{A})}L^2(\mathcal{A})_{C_0(\mathcal{A}^0)}$. The image of $C^*(\mathcal{A})$ on $\mathcal{L}(L^2(\mathcal{A}))$ is then isomorphic to $C_r^*(\mathcal{B})$, or in other words, the kernel of the map $C^*(\mathcal{A}) \to \mathcal{L}(L^2(\mathcal{A}))$ is the same as the kernel of the canonical map $\Lambda : C^*(\mathcal{A}) \to C_r^*(\mathcal{A})$.

Chapter 5

Groupoids, Fell bundles and associated *-algebras

In this chapter we present the basic set up which will enable us to define crossed products by Hecke pairs later in Chapter 6.

Our construction of a (*-algebraic) crossed product $A \times^{alg} G/\Gamma$ of an algebra A by a Hecke pair (G, Γ) will make sense when A is a certain algebra of sections of a Fell bundle over a discrete groupoid. In this chapter we show in detail what type of algebras A are involved in the crossed product and how they are obtained.

5.1 Group actions on discrete groupoids

Throughout this section G will denote a discrete group.

Definition 5.1.1. Let X be a groupoid. A *right action* of G on X is a mapping

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto xg \, , \end{aligned}$$

which is a right action of G on the underlying set of X, meaning that

1) $xe = x$, for all $x \in X$,

2) $x(g_1 g_2) = (xg_1)g_2$, for all $x \in X$, $g_1, g_2 \in G$,

which is compatible with the groupoid operations, meaning that

3) if x and y are composable in X, then so are xg and yg, for all $g \in G$, and moreover

$$(xg)(yg) = (xy)g\,,$$

4) $(xg)^{-1} = x^{-1}g$, for all $x \in X$ and $g \in G$.

Lemma 5.1.2. *Let X be a groupoid endowed with a right G-action. For every $x \in X$ and $g \in G$ we have*

$$\mathbf{s}(xg) = \mathbf{s}(x)g \qquad \textit{and} \qquad \mathbf{r}(xg) = \mathbf{r}(x)g\,.$$

In particular, G restricts to an action on the unit space X^0.

Proof: It follows easily from the definition of a right G-action that

$$\mathbf{s}(x)g = (x^{-1}x)g = (x^{-1}g)(xg) = (xg)^{-1}(xg) = \mathbf{s}(xg)\,,$$

and similarly for the range function. $\square$

Remark 5.1.3. Given elements x, y in a groupoid X endowed with a right G-action and given $g \in G$, we will often drop the brackets in expressions like $(xg)y$ and simply use the notation xgy. No confusion arises from this since G is only assumed to act on the right. On the other hand, we will never write an expression like xyg without brackets, since it can be confusing on whether it means $x(yg)$ or $(xy)g$.

The following definition is crucial for our definition of crossed products by Hecke pairs.

Definition 5.1.4. Let X be a groupoid endowed with a right G-action. A Fell bundle $\mathcal{A}$ over X is said to have *G-invariant fibers* if

$$\mathcal{A}_x = \mathcal{A}_{xg}$$

for all $x \in X$ and $g \in G$.

Proposition 5.1.5. *Let X be a groupoid endowed with a right G-action and $\mathcal{A}$ a Fell bundle with G-invariant fibers. The G-action on X gives rise to a well-defined action $\alpha : G \to Aut(C_c(\mathcal{A}))$ of G on $C_c(\mathcal{A})$ given by*

$$\alpha_g(f)\,(x) := f(xg)\,,$$

for $g \in G$, $f \in C_c(\mathcal{A})$ and $x \in X$.

Proof: Let us first prove that the action is well-defined, i.e. $\alpha_g(f) \in C_c(\mathcal{A})$. The fact that $\alpha_g(f)$ is finitely supported is obvious, so the only thing one needs to check is that $\alpha_g(f)$ is indeed a section of the bundle, i.e. $f(xg) \in \mathcal{A}_x$ for all $x \in X$, and this follows from the G-invariance of the fibers.

Let us now check that α_g is indeed a *-homomorphism for all $g \in G$. Linearity of α_g is obvious. Let $f, f_1, f_2 \in C_c(\mathcal{A})$. We have

$$\begin{aligned}
\alpha_g(f_1 \cdot f_2)\,(x) &= (f_1 \cdot f_2)\,(xg) \\
&= \sum_{\substack{y,z \in X \\ yz = xg}} f_1(y) f_2(z) \\
&= \sum_{\substack{y,z \in X \\ (yg^{-1})(zg^{-1}) = x}} f_1(y) f_2(z) \\
&= \sum_{\substack{y,z \in X \\ yz = x}} f_1(yg) f_2(zg) \\
&= \sum_{\substack{y,z \in X \\ yz = x}} \alpha_g(f_1)(y)\alpha_g(f_2)(z) \\
&= \big(\alpha_g(f_1)\alpha_g(f_2)\big)\,(x)\,.
\end{aligned}$$

Hence, $\alpha_g(f_1 \cdot f_2) = \alpha_g(f_1) \cdot \alpha_g(f_2)$. Also,

$$\begin{aligned}
\alpha_g(f^*)\,(x) &= f^*(xg) = \big(f(x^{-1}g)\big)^* \\
&= \big(\alpha_g(f)\,(x^{-1})\big)^* = \big(\alpha_g(f)\big)^*\,(x)\,.
\end{aligned}$$

Hence, $\alpha_g(f^*) = (\alpha_g(f))^*$. The fact that $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$ for every $g_1, g_2 \in G$ is also easily checked. □

Definition 5.1.6. Let X be a groupoid endowed with a right G-action and let H be a subgroup of G. We will say that the G-action is *H-good* if

$$\mathbf{s}(x)h = \mathbf{s}(x) \implies xh = x, \tag{5.1}$$

for $x \in X$ and $h \in H$.

We now give equivalent definitions of a H-good action. For that we recall from (1.9) that given an action of G on a set X we denote by $\mathcal{S}_x$ the stabilizer of the point $x \in X$.

Proposition 5.1.7. *Let X be a groupoid endowed with a right G-action and let H be a subgroup of G. The following statements are equivalent:*

i) The action is H-good.

ii) For any $x \in X$ we have

$$\mathcal{S}_{\mathbf{s}(x)} \cap H = \mathcal{S}_x \cap H = \mathcal{S}_{\mathbf{r}(x)} \cap H \,. \tag{5.2}$$

iii) The stabilizers of the H-action are the same on composable pairs, i.e. if $x \in X$ and $y \in Y$ are composable, then $\mathcal{S}_x \cap H = \mathcal{S}_y \cap H$.

Proof: $i) \Longrightarrow ii)$ Since the action is H-good we have, by definition, that $\mathcal{S}_{\mathbf{s}(x)} \cap H \subseteq \mathcal{S}_x \cap H$. Also, if $h \in \mathcal{S}_x \cap H$, then we have $xh = x$, and therefore by Lemma 5.1.2 we get $\mathbf{s}(x) = \mathbf{s}(xh) = \mathbf{s}(x)h$, from which we conclude that $h \in \mathcal{S}_{\mathbf{s}(x)} \cap H$. Hence we have $\mathcal{S}_{\mathbf{s}(x)} \cap H = \mathcal{S}_x \cap H$.

Since we have that $(xg)^{-1} = x^{-1}g$, it follows easily that $\mathcal{S}_x = \mathcal{S}_{x^{-1}}$. Hence, from this observation and what we proved above it follows that

$$\mathcal{S}_{\mathbf{r}(x)} \cap H \quad = \quad \mathcal{S}_{\mathbf{s}(x^{-1})} \cap H \quad = \quad \mathcal{S}_{x^{-1}} \cap H \quad = \quad \mathcal{S}_x \cap H \,.$$

Thus we have proven that $i)$ implies $ii)$.

$ii) \Longrightarrow iii)$ Suppose $x \in X$ and $y \in X$ are composable. Then, from $ii)$ we have that

$$\mathcal{S}_x \cap H = \mathcal{S}_{\mathbf{s}(x)} \cap H = \mathcal{S}_{\mathbf{r}(y)} \cap H = \mathcal{S}_y \cap H \,.$$

$iii) \Longrightarrow i)$ Assume now that the stabilizers of the H-action are the same on composable elements. Since x and $\mathbf{s}(x)$ are composable we have that $\mathcal{S}_x \cap H = \mathcal{S}_{\mathbf{s}(x)} \cap H$, which implies that the action is H-good. $\square$

It is easy to see that any H-good action is also gHg^{-1}-good for any conjugate gHg^{-1}, and also K-good for any subgroup $K \subseteq H$.

The following property will also be important for defining crossed products by Hecke pairs:

Definition 5.1.8. Let X be a groupoid endowed with a right G-action and let H be a subgroup of G. We will say that the action has the *H-intersection property* if

$$uH \cap ugHg^{-1} = uH^g \,, \tag{5.3}$$

for every unit $u \in X^0$ and $g \in G$.

We defer examples of H-good actions and actions with the H-intersection property for the next section. We now introduce one of the important ingredients for our definition of crossed products by Hecke pairs: the orbit space groupoid.

Let G be a group, $H \subseteq G$ a subgroup and X a groupoid endowed with a H-good right G-action. Then, the orbit space X/H becomes a groupoid in a canonical way which we will now describe. For that, and throughout this text, we will use the following notation: given elements x, y we define the set

$$H_{x,y} := \{h \in H : \mathbf{s}(x)h = \mathbf{r}(y)\} \,. \tag{5.4}$$

The groupoid structure on X/H is described as follows:

- A pair $(xH, yH) \in (X/H)^2$ is composable if and only if $H_{x,y} \neq \emptyset$, or equivalently, $\mathbf{r}(y) \in \mathbf{s}(x)H$. This property is easily seen not to depend on the choice of representatives x, y from the orbits xH, yH respectively.
- Given a composable pair $(xH, yH) \in (X/H)^2$, their product is

  $$xH\, yH := x\widetilde{h}yH \,, \tag{5.5}$$

  where $\widetilde{h}$ is any element of $H_{x,y}$. It will follow from the fact the action is H-good that $x\widetilde{h}$ does not depend on the representative $\widetilde{h}$ chosen from $H_{x,y}$. The result of the product $xH\, yH$ also does not depend on the choice of representatives x, y. We will prove this in the next result.
- The inverse of the element xH is simply the element $x^{-1}H$. It is also easy to see that this does not depend on the choice of representative x.

Proposition 5.1.9. *Let G be a group, $H \subseteq G$ a subgroup and X a groupoid endowed with a H-good right G-action. The operations above give the orbit space X/H the structure of a groupoid. Moreover, the unit space $(X/H)^0$ of* *this groupoid is $X^0/H = \{uH : u \in X^0\}$, where X^0 is the unit space of X, and the range and source functions satisfy*

$$\mathbf{s}(xH) = \mathbf{s}(x)H \qquad \textit{and} \qquad \mathbf{r}(xH) = \mathbf{r}(x)H\,.$$

Proof: Let us first prove that the product is well-defined. Let $(xH, yH) \in (X/H)^2$ be a composable pair. The fact that $x\widetilde{h}$ does not depend on the representative $\widetilde{h}$ chosen from $H_{x,y}$ follows from the assumption that the action is H-good, since if $h_1, h_2 \in H_{x,y}$ then we have

$$\mathbf{s}(x)h_1 = \mathbf{r}(y) = \mathbf{s}(x)h_2\,,$$

and therefore $\mathbf{s}(x)h_1h_2^{-1} = \mathbf{s}(x)$, and because the action is H-good $xh_1h_2^{-1} = x$, i.e. $xh_1 = xh_2$.

Let us now prove that X/H is a groupoid with the operations above. We check associativity first. Suppose $xH, yH, zH \in X/H$ are such that (xH, yH) is composable and (yH, zH) is composable. We want to prove that $(xHyH, zH)$ and $(xH, yHzH)$ are also composable and moreover $(xHyH)zH = xH(yHzH)$. We have by definition that $xHyH = x\widetilde{h_1}yH$ and $yHzH = y\widetilde{h_2}zH$, where $\widetilde{h_1}$ is any element of $H_{x,y}$ and $\widetilde{h_2}$ is any element of $H_{y,z}$. We now notice that

$$H_{x\widetilde{h_1}y,z} = \{h \in H : \mathbf{s}(x\widetilde{h_1}y)h = \mathbf{r}(z)\} = \{h \in H : \mathbf{s}(y)h = \mathbf{r}(z)\} = H_{y,z}\,.$$

Since $H_{y,z} \neq \emptyset$ it follows that $H_{x\widetilde{h_1}y,z} \neq \emptyset$, and therefore $(x\widetilde{h_1}yH, zH)$ is composable. Similarly,

$$\begin{aligned} H_{x,y\widetilde{h_2}z} &= \{h \in H : \mathbf{s}(x)h = \mathbf{r}(y\widetilde{h_2}z)\} \\ &= \{h \in H : \mathbf{s}(x)h = \mathbf{r}(y)\widetilde{h_2}\} \\ &= \{h \in H : \mathbf{s}(x)h\widetilde{h_2}^{-1} = \mathbf{r}(y)\} \\ &= H_{x,y}\,\widetilde{h_2}\,. \end{aligned}$$

Hence, since $H_{x,y} \neq \emptyset$ it follows that $H_{x,y\widetilde{h_2}z} \neq \emptyset$, and therefore $(xH, y\widetilde{h_2}zH)$ is composable.

As we saw above $H_{x\widetilde{h_1}y,z} = H_{y,z}$, and since $\widetilde{h_2} \in H_{y,z}$, we can write

$$\begin{aligned} (xHyH)zH &= x\widetilde{h_1}yHzH = (x\widetilde{h_1}y)\widetilde{h_2}zH \\ &= x\widetilde{h_1}\widetilde{h_2}y\widetilde{h_2}zH\,. \end{aligned}$$

Also seen above, we have that $H_{x,y\widetilde{h_2}z} = H_{x,y}\,\widetilde{h_2}$, so that $\widetilde{h_1}\widetilde{h_2} \in H_{x,y\widetilde{h_2}z}$. Hence, we conclude that

$$(xHyH)zH \quad = \quad xH(yHzH)\,.$$

We now check that for any element $xH \in X/H$ we have that $(xH, x^{-1}H)$ and $(x^{-1}H, xH)$ are composable pairs. We have that

$$H_{x,x^{-1}} \quad = \quad \{h \in H : \mathbf{s}(x)h = \mathbf{r}(x^{-1})\} \quad = \quad \{h \in H : \mathbf{s}(x)h = \mathbf{s}(x)\}\,,$$

and the identity element e obviously belongs to the latter set. Hence we conclude that $H_{x,x^{-1}} \neq \emptyset$, and therefore $(xH, x^{-1}H)$ is composable. A similar observation shows that $(x^{-1}H, xH)$ is also composable.

To prove that X/H is a groupoid it now remains to prove the inverse identities $xHyHy^{-1}H = xH$ and $y^{-1}HyHxH = xH$, in case (xH, yH) is composable (for the first identity) and (yH, xH) is composable (for the second identity). We first show that $yHy^{-1}H = \mathbf{r}(y)H$. We have that $yHy^{-1}H = y\widetilde{h}y^{-1}H$ for any element $\widetilde{h} \in H_{y,y^{-1}}$. Since, as we observed above, we always have $e \in H_{y,y^{-1}}$, it follows that we can take $\widetilde{h}$ as e. Thus, we get

$$yHy^{-1}H \quad = \quad yy^{-1}H \quad = \quad \mathbf{r}(y)H\,. \tag{5.6}$$

From this it follows that

$$xHyHy^{-1}H \quad = \quad xH\mathbf{r}(y)H \quad = \quad x\widetilde{h_1}\mathbf{r}(y)H\,,$$

where $\widetilde{h_1}$ is any element of $H_{x,\mathbf{r}(y)}$. By definition, $\widetilde{h_1}$ is such that $\mathbf{r}(y) = \mathbf{s}(x)\widetilde{h_1} = \mathbf{s}(x\widetilde{h_1})$. Hence we have that $x\widetilde{h_1}\mathbf{r}(y) = x\widetilde{h_1}$, and therefore

$$xHyHy^{-1}H = x\widetilde{h_1}H = xH\,.$$

The other identity $y^{-1}HyHxH = xH$ is proven in a similar fashion. Hence, we conclude that X/H is a groupoid.

From equality (5.6) it follows easily that the units of X/H are precisely the elements of the form uH where $u \in X^0$, so that we can write $(X/H)^0 = X^0/H$. Also from (5.6) it follows that the range function in X/H satisfies:

$$\mathbf{r}(xH) = \mathbf{r}(x)H\,.$$

The analogous result for the source function is proven in a similar fashion. $\square$

Let X be a groupoid endowed with a H-good right G-action, where H is a subgroup of G. Given a Fell bundle $\mathcal{A}$ over X with G-invariant fibers, we naturally get a Fell bundle $\mathcal{A}/H$ over the groupoid X/H, whose fibers are

$$\big(\mathcal{A}/H\big)_{xH} := \mathcal{A}_x\,. \tag{5.7}$$

Of course, this is possible because of the G-invariance of the fibers, so that $\mathcal{A}_x = \mathcal{A}_{xh}$ for any $h \in H$.

5.2 Examples

In this section we give some examples of H-good actions and actions satisfying the H-intersection property. For the rest of the section X denotes a groupoid endowed with a right G-action and $H \subseteq G$ denotes a subgroup.

The first two examples (5.2.1 and 5.2.2) show that H-good actions that satisfy the H-intersection property are present in actions that have completely opposite behaviours, such as free actions and actions that fix every point.

Example 5.2.1. If the restricted action of H on the united space X^0 is free, then the action is H-good and satisfies the H-intersection property.

Example 5.2.2. If the restricted action of H on the united space X^0 fixes every point, then the action is H-good and satisfies the H-intersection property.

In the case groupoid structure is trivial, i.e. when the groupoid is just a set, then the condition that defines a H-good action becomes trivial:

Example 5.2.3. If X is a set, then the action is automatically H-good.

The following example is the motivating example for the Katayama duality for Echterhoff-Quigg crossed product, that will be discussed in more detail in Chapter 11.

Example 5.2.4. Suppose X is the transformation groupoid $G \times G$. We recall that the multiplication and inversion operations on this groupoid are given by:

$$(s, tr)(t, r) = (st, r) \qquad \text{and} \qquad (s, t)^{-1} = (s^{-1}, st)\,.$$

Recall also that the source and range functions on $G \times G$ are defined by

$$\mathbf{s}(s, t) = (e, t) \qquad \text{and} \qquad \mathbf{r}(s, t) = (e, st)\,.$$

We observe that there is a natural right G-action on $G \times G$, given by

$$(s, t)g := (s, tg)\,.$$

Let δ be a coaction of G on a C^*-algebra B and $\mathcal{B}$ the associated Fell bundle. Following [11, Section 3], we will denote by $\mathcal{A} := \mathcal{B} \times G$ the corresponding Fell bundle over the groupoid $G \times G$, whose fibers are

$$(\mathcal{B} \times G)_{(s,t)} := \mathcal{B}_s \,.$$

It is then clear that $\mathcal{A} = \mathcal{B} \times G$ has G-invariant fibers.

The G-action on $G \times G$ is free and therefore is H-good and satisfies the H-intersection property with respect to any subgroup $H \subseteq G$. The orbit groupoid $(G \times G)/H$ can be canonically identified with the groupoid $G \times G/H$ of [10], whose operations are given by:

$$(s, trH)(t, rH) = (st, rH) \qquad \text{and} \qquad (s, tH)^{-1} = (s^{-1}, stH)\,.$$

Moreover, the Fell bundle $\mathcal{A}/H$ is canonically identified with the Fell bundle $\mathcal{B} \times G/H$ over $G \times G/H$ defined in [10], and in this way $C_c(\mathcal{A}/H)$ is canonically isomorphic with the Echterhoff-Quigg algebra $C_c(\mathcal{B} \times G/H)$, also from [10].

5.3 The algebra $M(C_c(\mathcal{A}))$

We will assume for the rest of this section that $H \subseteq G$ is any subgroup, X is a groupoid endowed with a H-good right G-action and $\mathcal{A}$ is a Fell bundle with G-invariant fibers. We recall that $\mathcal{A}/H$ stands for the associated Fell bundle over the groupoid X/H, as defined in (5.7).

For the purpose of defining crossed products by Hecke pairs it is convenient to have a "large" algebra which contains the algebras $C_c(\mathcal{A}/H)$ for different subgroups $H \subseteq G$. In this way we are allowed to multiply elements of $C_c(\mathcal{A}/H)$ and $C_c(\mathcal{A}/K)$, for different subgroups $H, K \subseteq G$, in a meaningful way. This large algebra will be the multiplier algebra $M(C_c(\mathcal{A}))$. This section is thus devoted to show how algebras such as $C_c(\mathcal{A}/H)$ and $C_c(X^0/H)$ embed in $M(C_c(\mathcal{A}))$ in a canonical way.

Our first result shows that there is a natural inclusion $C_c(\mathcal{A}/H) \subseteq M(C_c(\mathcal{A}))$.

Theorem 5.3.1. *There is an embedding ι of $C_c(\mathcal{A}/H)$ into $M(C_c(\mathcal{A}))$ determined by the following rule: for any $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$ we have*

$$\iota(a_{xH})b_y := \begin{cases} (ab)_{x\tilde{h}y}\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \tag{5.8}$$

where $\widetilde{h}$ is any element of $H_{x,y}$.

Remark 5.3.2. The above result allows us to see $C_c(\mathcal{A}/H)$ as a *-subalgebra of $M(C_c(\mathcal{A}))$. We shall henceforward drop the symbol ι and make no distinction of notation between an element of $C_c(\mathcal{A}/H)$ and its correspondent multiplier in $M(C_c(\mathcal{A}))$.

Proof of Theorem 5.3.1: Let us first show that expression (5.8) does indeed define an element of $M(C_c(\mathcal{A}))$. For this it is enough to check that $\langle \iota(a_{xH})b_y, c_z\rangle = \langle b_y, \iota(a^*_{x^{-1}H})\, c_z\rangle$, for all $b \in \mathcal{A}_y$ and $c \in \mathcal{A}_z$, with $y, z \in X$. For $\iota(a_{xH})b_y$ to be non-zero, we must necessarily have $H_{x,y} \neq \emptyset$, and in this case $\iota(a_{xH})b_y = (ab)_{x\widetilde{h}y}$, where $\widetilde{h} \in H_{x,y}$. Now,

$$\begin{aligned}\langle \iota(a_{xH})b_y, c_z\rangle &= \langle (ab)_{x\widetilde{h}y}, c_z\rangle = (b^*a^*)_{y^{-1}(x^{-1}\widetilde{h})}c_z \\ &= b^*_{y^{-1}}a^*_{x^{-1}\widetilde{h}}\, c_z\end{aligned}$$

For $a^*_{x^{-1}\widetilde{h}}\, c_z$ to be non-zero we must necessarily have $\mathbf{r}(z) = \mathbf{s}(x^{-1})\widetilde{h}$, i.e. $\widetilde{h} \in H_{x^{-1},z}$. So, to summarize, for $\langle a_{xH}b_y, c_z\rangle$ to be non-zero we must have $H_{x,y} \cap H_{x^{-1},z} \neq \emptyset$ and in this case we obtain

$$\langle \iota(a_{xH})b_y, c_z\rangle = b^*_{y^{-1}}a^*_{x^{-1}\widetilde{h}}\, c_z\,,$$

where $\widetilde{h}$ is any element of $H_{x,y} \cap H_{x^{-1},z}$. A similar computation for $\langle b_y, \iota(a^*_{x^{-1}H})\, c_z\rangle$ yelds the exact same result.

Recall from (4.3) that any $f \in C_c(\mathcal{A}/H)$ can be written as

$$f = \sum_{xH \in X/H} \big(f(xH)\big)_{xH}\,.$$

From this we are able to define a multiplier $\iota(f) \in M(C_c(\mathcal{A}))$, simply by extending expression (5.8) by linearity.

We want to show that ι is an injective *-homomorphism. First, we claim that given $a_{xH}, b_{yH} \in C_c(\mathcal{A}/H)$ we have

$$\iota(a_{xH})\iota(b_{yH}) = \iota(a_{xH}b_{yH})\,.$$

This amounts to proving that

$$\iota(a_{xH})\iota(b_{yH}) = \begin{cases} \iota((ab)_{x\widetilde{h}yH})\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise}\end{cases}$$

with $\widetilde{h}$ being any element of $H_{x,y}$. To see this, let $c_z \in \mathcal{A}_z$, with $z \in X$. We have

$$\begin{aligned}
\iota(a_{xH})\iota(b_{yH})c_z &= \begin{cases} \iota(a_{xH})(bc)_{yh_0z}\,, & \text{if } \ H_{y,z} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (abc)_{xh_1yh_0z}\,, & \text{if } \ H_{y,z} \neq \emptyset \ \text{ and } \ H_{x,yh_0z} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

with $h_0 \in H_{y,z}$ and $h_1 \in H_{x,yh_0z}$. But $H_{x,yh_0z} = H_{x,yh_0} = H_{x,y}\,h_0$, hence the above can be written as

$$\begin{aligned}
&= \begin{cases} (abc)_{x\widetilde{h}h_0yh_0z}\,, & \text{if } \ H_{y,z} \neq \emptyset \ \text{ and } \ H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (abc)_{(x\widetilde{h}y)h_0z}\,, & \text{if } \ H_{y,z} \neq \emptyset \ \text{ and } \ H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

where $\widetilde{h} \in H_{x,y}$. Also, $H_{y,z} = H_{x\widetilde{h}y,z}$. Thus, we obtain

$$\begin{aligned}
&= \begin{cases} (abc)_{(x\widetilde{h}y)h_0z}\,, & \text{if } \ H_{x\widetilde{h}y,z} \neq \emptyset \ \text{ and } \ H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \iota((ab)_{x\widetilde{h}yH})\,c_z\,, & \text{if } \ H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Since ι is linear and multiplicative on the elements of the form a_{xH}, it is necessarily an homomorphism. Now the fact that $\iota(a_{xH})^* = \iota((a_{xH})^*) = \iota(a^*_{x^{-1}H})$ follows directly from the computations in the beginning of this proof. Hence, ι is a *-homomorphism.

Let us now prove injectivity of ι. Suppose $f \in C_c(\mathcal{A}/H)$ is such that $\iota(f) = 0$. Decomposing f as a sum of elements of the form a_{xH} we get

$$\sum_{xH \in X/H} \iota\Big(\big(f(xH)\big)_{xH}\Big) = 0\,.$$

For any $y \in X$ we then have

$$\begin{aligned}
0 &= \sum_{xH \in X/H} \iota\Big(\big(f(xH)\big)_{xH}\Big)\big(f(yH)^*\big)_{y^{-1}} \\
&= \sum_{\substack{xH \in X/H \\ \mathbf{s}(y) \in \mathbf{s}(x)H}} \iota\Big(\big(f(xH)\big)_{xH}\Big)\big(f(yH)^*\big)_{y^{-1}} \\
&= \sum_{\substack{xH \in X/H \\ \mathbf{s}(y) \in \mathbf{s}(x)H}} \big(f(xH)f(yH)^*\big)_{x\widetilde{h_x}y^{-1}}\,,
\end{aligned}$$

where $\widetilde{h_x}$ is any element of $H_{x,y^{-1}}$. Now the elements $x\widetilde{h_x}y^{-1}$ in the sum above are all different, because if we had $x_1\widetilde{h_{x_1}}y^{-1} = x_2\widetilde{h_{x_2}}y^{-1}$, then we would have $x_1\widetilde{h_{x_1}} = x_2\widetilde{h_{x_2}}$ and therefore $x_1H = x_2H$. Therefore each of the summands in the above sum is zero, and in particular we must have

$$\begin{aligned} 0 &= \big(f(yH)f(yH)^*\big)_{y\widetilde{h_y}y^{-1}} \\ &= \big(f(yH)f(yH)^*\big)_{\mathbf{r}(y)}, \end{aligned}$$

and therefore $f(yH)f(yH)^* = 0$. Since $f(yH)$ is an element of a C^*-algebra, we get $f(yH) = 0$, and since this is true for any $y \in X$, we have $f = 0$, i.e. ι is injective. □

Proposition 5.3.3. *There is an embedding ι of $C_b(X^0)$ into $M(C_c(\mathcal{A}))$ defined by*

$$\iota(f)\, b_y := f(\mathbf{r}(y))b_y\,. \tag{5.9}$$

for every $f \in C_b(X^0)$, $y \in X$ and $b \in \mathcal{A}_y$.

Remark 5.3.4. The above result allows us to see $C_b(X^0)$ as a *-subalgebra of $M(C_c(\mathcal{A}))$. We shall henceforward drop the symbol ι and make no distinction of notation between an element of $C_b(X^0)$ and its correspondent multiplier in $M(C_c(\mathcal{A}))$.

Proof of Proposition 5.3.3 : It is easy to see that $\langle \iota(f)b_y\, ,\, c_z\rangle = \langle b_y\, ,\, \iota(f^*)c_z\rangle$ for any $y, z \in X$, $b \in \mathcal{A}_y$ and $c \in \mathcal{A}_z$, so that the expression (5.9) does define an element of $M(C_c(\mathcal{A}))$.

Hence we get a linear map $\iota : C_b(X^0) \to M(C_c(\mathcal{A}))$. Given two elements $f_1, f_2 \in C_b(X^0)$, we have that

$$\iota(f_1)\iota(f_2)b_y = f_1(\mathbf{r}(y))f_2(\mathbf{r}(y))b_y = \iota(f_1f_2)b_y$$

for any $y \in X$ and$b \in \mathcal{A}_y$, so that ι is a *-homomorphism. Hence, we only need to prove that ι is injective. This is not difficult to see: given $f \in C_b(X^0)$ such that $\iota(f) = 0$ we have, for any unit $u \in X^0$ and $b \in \mathcal{A}_u$, that

$$0 \;=\; \iota(f)b_u \;=\; f(u)b_u\,.$$

Hence $f(u) = 0$, because each fiber $\mathcal{A}_u$ is non-zero by definition of a Fell bundle. Since this is true for any $u \in X^0$ we get $f = 0$, i.e. ι is injective. □

Recall, from Lemma 5.1.2, that the action of G on X restricts to an action of G on the set X^0. Thus it makes sense to talk about the commutative *-algebra

$$C_c(X^0/H) \subseteq C_b(X^0)\,.$$

Since there is a canonical embedding, given by Proposition 5.3.3, of $C_b(X^0)$ into $M(C_c(\mathcal{A}))$, we have in particular an embedding of $C_c(X^0/H)$ into $M(C_c(\mathcal{A}))$ which identifies an element $f \in C_c(X^0/H)$ with the multiplier $f \in M(C_c(\mathcal{A}))$ given by:

$$fb_y := f(\mathbf{r}(y)H)b_y\,.$$

Moreover Proposition 5.3.3 applied to the groupoid X/H and the Fell bundle $\mathcal{A}/H$ shows that there is a canonical embedding of $C_b(X^0/H)$ into $M(C_c(\mathcal{A}/H))$, which identifies an element $f \in C_b(X^0/H)$ with the multiplier $f \in M(C_c(\mathcal{A}/H))$ given by

$$fb_{yH} := f(\mathbf{r}(y)H)b_{yH}\,. \tag{5.10}$$

Since both $C_c(X^0/H)$ and $C_c(\mathcal{A}/H)$ are canonically embedded in $M(C_c(\mathcal{A}))$, it is convenient to understand what happens (inside $M(C_c(\mathcal{A}))$) when one multiplies an element of $C_c(X^0/H)$ by an element $C_c(\mathcal{A}/H)$. Perhaps unsurprisingly, this product is given exactly by expression (5.10), which models the action of $C_c(X^0/H)$ on $C_c(\mathcal{A}/H)$ as multipliers of the latter algebra. In other words, it makes no difference to view $C_c(X^0/H)$ inside $M(C_c(\mathcal{A}/H))$ or inside $M(C_c(\mathcal{A}))$ when it comes to multiplication by elements of $C_c(\mathcal{A}/H)$.

We will now show how the multiplication of elements of $C_c(\mathcal{A}/H)$ by elements of $C_c(X^0)$ is determined (inside $M(C_c(\mathcal{A}))$). Before we proceed we will first introduce some notation that will be used throughout this work: Given a set $A \subset X^0$ we will denote by $1_A \in C_b(X^0)$ the characteristic function of A. In case A is a singleton $\{u\}$ we will simply write 1_u.

Proposition 5.3.5. *Inside $M(C_c(\mathcal{A}))$ we have that, for $x \in X$, $a \in \mathcal{A}_x$ and $u \in X^0$,*

$$a_{xH}1_u = \begin{cases} a_{x\widetilde{h}}\,, & \text{if } H_{x,u} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

where $\widetilde{h}$ is any element of $H_{x,u}$.

Proof: Let $y \in Y$ and $b \in \mathcal{A}_y$. For the product $a_{xH} 1_u\, b_y$ to be non-zero we must necessarily have $u = \mathbf{r}(y)$ (from (5.9)), and in this case we obtain

$$a_{xH} 1_u\, b_y = a_{xH} b_y = (ab)_{x\widetilde{h}y} = a_{x\widetilde{h}} b_y\,,$$

where $\widetilde{h}$ is any element of $H_{x,y}$. Since $u = \mathbf{r}(y)$, we have $H_{x,y} = H_{x,u}$, and this concludes the proof. $\square$

It will be of particular importance to know how to multiply, inside $M(C_c(\mathcal{A}))$, elements of $C_c(\mathcal{A}/H)$ with elements of $C_c(\mathcal{A}/K)$ when $K \subseteq H$ is an arbitrary subgroup. It turns out that the algebra $C_c(\mathcal{A}/K)$ is preserved by multiplication by elements of $C_c(\mathcal{A}/H)$, as we show in the next result:

Proposition 5.3.6. *Let $K \subseteq H$ be any subgroup. We have that*

$$a_{xH} b_{yK} = \begin{cases} (ab)_{x\widetilde{h}yK}\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \tag{5.11}$$

where $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$. In particular $C_c(\mathcal{A}/K)$ is invariant under multiplication by elements of $C_c(\mathcal{A}/H)$.

Proof: First we observe that since the action is assumed to be H-good, it is automatically K-good, so that we can form the groupoid X/K and the Fell bundle $\mathcal{A}/K$.

Let $z \in X$ and $c \in \mathcal{A}_z$. We have that

$$\begin{aligned} a_{xH} b_{yK} c_z &= \begin{cases} a_{xH} (bc)_{y\widetilde{k}z}\,, & \text{if } K_{y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} (abc)_{x\widetilde{h}y\widetilde{k}z}\,, & \text{if } H_{x,y\widetilde{k}z} \neq \emptyset \text{ and } K_{y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} (abc)_{\left(x\widetilde{h}\widetilde{k}^{-1}y\right)\widetilde{k}z}\,, & \text{if } H_{x,y\widetilde{k}z} \neq \emptyset \text{ and } K_{y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\widetilde{k}$ is any element of $K_{y,z}$ and $\widetilde{h}$ is any element of $H_{x,y\widetilde{k}z}$. Now, since $H_{x,y\widetilde{k}z} = H_{x,y\widetilde{k}} = H_{x,y}\widetilde{k}$, it follows that $\widetilde{h}\widetilde{k}^{-1} \in H_{x,y}$, and moreover since

$K_{y,z} = K_{x\widetilde{h}\widetilde{k}^{-1}y,z}$, we conclude that

$$= \begin{cases} (abc)_{\left(x\widetilde{h}\widetilde{k}^{-1}y\right)\widetilde{k}z}\,, & \text{if } H_{x,y} \neq \emptyset \text{ and } K_{x\widetilde{h}\widetilde{k}^{-1}y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} (ab)_{x\widetilde{h}\widetilde{k}^{-1}yK}c_z\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Thus (5.11) follows immediately (the element $\widetilde{h}$ in (5.11) is simply the element denoted by $\widetilde{h}\widetilde{k}^{-1}$ above). □

In case the subgroup K has finite index in H we can strengthen Proposition 5.3.6 in the following way:

Proposition 5.3.7. *Let $K \subseteq H$ be a subgroup such that $[H : K] < \infty$. Inside $M(C_c(\mathcal{A}))$ we have that*

$$a_{xH} = \sum_{[h]\in\mathcal{S}_x\backslash H/K} a_{xhK}\,, \tag{5.12}$$

for any $x \in X$ and $a \in \mathcal{A}_x$. In particular, inside $M(C_c(\mathcal{A}))$ we have that $C_c(\mathcal{A}/H)$ is a *-subalgebra of $C_c(\mathcal{A}/K)$.

Proof: First we notice that since $[H : K] < \infty$ we have that the right hand side of (5.12) is a finite sum and therefore does indeed define an element of $C_c(\mathcal{A}/K)$. To prove this result it suffices to show that

$$a_{xH}b_y = \sum_{[h]\in\mathcal{S}_x\backslash H/K} a_{xhK}b_y\,, \tag{5.13}$$

for all $y \in X$ and $b \in \mathcal{A}_y$. First we notice that both the right and left hand sides of (5.13) are zero unless $\mathbf{r}(y) \in \mathbf{s}(x)H$. In case $\mathbf{r}(y) \in \mathbf{s}(x)H$ we have

$$a_{xH}b_y = (ab)_{x\widetilde{h}y}\,,$$

where $\widetilde{h}$ is any element of $H_{x,y}$.

Recall from Proposition 1.2.2 that there is a bijective correspondence between the set of K-orbits $(xH)/K$ and the double coset space $\mathcal{S}_x\backslash H/K$. It is clear that $a_{x\widetilde{h}K}b_y = (ab)_{x\widetilde{h}y}$. Moreover, for all the classes $[h] \neq [\widetilde{h}]$ in

$\mathcal{S}_x\backslash H/K$ we have $\mathbf{r}(y) \notin \mathbf{s}(x)hK$, because $\mathbf{r}(y) \in \mathbf{s}(x)\widetilde{h}K$. Hence, for all the classes $[h] \neq [\widetilde{h}]$ in $\mathcal{S}_x\backslash H/K$ we have $a_{xhK}b_y = 0$. We conclude that

$$\sum_{[h]\in\mathcal{S}_x\backslash H/K} a_{xhK}b_y = (ab)_{x\widetilde{h}y}\,,$$

and equality (5.13) is proven. □

Remark 5.3.8. In Proposition 5.3.7 the fact that $[H : K] < \infty$ was only used to ensure that the sum on the right hand side of (5.12) was finite. One could more generally just require that the sets $\mathcal{S}_x\backslash H/K$ are finite for all $x \in X$, but this generality will not be used here.

As we saw in Proposition 5.1.5 we have an action α of G on $C_c(\mathcal{A})$. This action can be extended in a unique way to an action on $M(C_c(\mathcal{A}))$, which we will still denote by α. We will now show what this action on $M(C_c(\mathcal{A}))$ does to the algebras $C_b(X^0)$, $C_c(\mathcal{A}/H)$ and $C_c(X^0/H)$.

Proposition 5.3.9. *The extension of the action α to $M(C_c(\mathcal{A}))$, also denoted by α, satisfies the following properties:*

(i) The restriction of α to $C_b(X^0)$ is precisely the action that comes from the G-action on X^0.

(ii) For any $g \in G$ the automorphism α_g takes $C_c(X^0/H)$ to $C_c(X^0/gHg^{-1})$, by

$$\alpha_g(1_{xH}) = 1_{(xg^{-1})(gHg^{-1})}\,. \tag{5.14}$$

(iii) For any $g \in G$ the automorphism α_g takes $C_c(\mathcal{A}/H)$ to $C_c(\mathcal{A}/gHg^{-1})$, by

$$\alpha_g(a_{xH}) = a_{(xg^{-1})(gHg^{-1})}\,. \tag{5.15}$$

(iv) Both $C_c(\mathcal{A}/H)$ and $C_c(X^0/H)$ are contained in $M(C_c(\mathcal{A}))^H$.

Proof: *(i)* Let $y \in X$, $b \in \mathcal{A}_y$ and $f \in C_b(X^0)$. For a $g \in G$ let us denote by $f_g \in C_b(X^0)$ the function defined by $f_g(x) = f(xg)$. By definition of the extension of α to $M(C_c(\mathcal{A}))$, we have

$$\begin{aligned} \alpha_g(f)\, b_y &= \alpha_g(f\alpha_g^{-1}(b_y)) &= \alpha_g(f b_{yg}) \\ &= \alpha_g(f(\mathbf{r}(yg))b_{yg}) &= \alpha_g(f(\mathbf{r}(y)g)b_{yg}) \\ &= f(\mathbf{r}(y)g)b_y &= f_g(\mathbf{r}(y))b_y \\ &= f_g b_y\,. \end{aligned}$$

Hence we conclude that $\alpha_g(f) = f_g$ and therefore the action α on $C_b(X^0)$ is just the action that comes from the G-action on X^0.

(ii) This follows directly from (i).

(iii) Let $y \in X$ and $b \in \mathcal{A}_y$. By definition of the extension of α to $M(C_c(\mathcal{A}))$, we have

$$\alpha_g(a_{xH})\, b_y = \alpha_g(a_{xH}\alpha_g^{-1}(b_y)) = \alpha_g(a_{xH}b_{yg})\,.$$

Also, we can see that

$$\begin{aligned} \alpha_g(a_{xH}b_{yg}) &= \begin{cases} \alpha_g\big((ab)_{x\widetilde{h}yg}\big)\,, & \text{if } H_{x,yg} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} (ab)_{x\widetilde{h}g^{-1}y}\,, & \text{if } H_{x,yg} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} (ab)_{xg^{-1}g\widetilde{h}g^{-1}y}\,, & \text{if } H_{x,yg} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $\widetilde{h} \in H_{x,yg}$. Now an easy computation shows that we have

$$H_{x,yg} = g^{-1}\big(gHg^{-1}\big)_{xg^{-1},y}\, g\,,$$

and thereby we obtain, for $t \in \big(gHg^{-1}\big)_{xg^{-1},y}$,

$$\begin{aligned} \alpha_g(a_{xH})\, b_y &= \begin{cases} (ab)_{xg^{-1}ty}\,, & \text{if } \big(gHg^{-1}\big)_{xg^{-1},y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= a_{(xg^{-1})(gHg^{-1})}\, b_y\,. \end{aligned}$$

(iv) This follows directly from (ii) and (iii). □

It is important to know how to multiply an element of $C_c(\mathcal{A}/H)$ with an element of $C_c(X^0/gHg^{-1})$ inside $M(C_c(\mathcal{A}))$. This is easy if we are under the assumption that G-action satisfies the H-intersection property:

Proposition 5.3.10. *Let X be a groupoid endowed with a H-good right G-action that satisfies the H-intersection property. Then, for every $x \in X$ and $g \in G$ the following equality hold in $M(C_c(\mathcal{A}))$:*

$$a_{xH}\, 1_{\mathbf{s}(x)gHg^{-1}} = a_{xH^g}\, .$$

Proof: For any $y \in X$ and $b \in \mathcal{A}_y$ we have

$$\begin{aligned}
a_{xH}\, 1_{\mathbf{s}(x)gHg^{-1}}\, b_y &= \begin{cases} a_{xH} b_y\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)gHg^{-1} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (ab)_{x\widetilde{h}y}\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)gHg^{-1} \text{ and } \mathbf{r}(y) \in \mathbf{s}(x)H \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (ab)_{x\widetilde{h}y}\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)H \cap \mathbf{s}(x)gHg^{-1} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

where $\widetilde{h} \in H_{x,y}$. Now, by the H-intersection property, we obtain

$$= \begin{cases} (ab)_{x\widetilde{h}y}\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)H^g \\ 0, & \text{otherwise} \end{cases}$$

Of course, we have $(H^g)_{x,y} \subseteq H_{x,y}$, and hence we can choose $\widetilde{h}$ as an element of $(H^g)_{x,y}$, thereby obtaining

$$= a_{xH^g}\, b_y\,,$$

which finishes the proof. □

Chapter 6

*-Algebraic crossed product by a Hecke pair

In this chapter we introduce our notion of a (*-algebraic) crossed product by a Hecke pair and we explore its basic properties and its representation theory. Throughout the rest of this work we impose the following standing assumption, based on the tools developed in Chapter 5 section 5.1.

Standing Assumption 6.0.11. We assume from now on that (G, Γ) is a Hecke pair, X is a groupoid with a Γ-good right G-action satisfying the Γ-intersection property, $\mathcal{A}$ is a Fell bundle over X with G-invariant fibers and α is the corresponding action of G on $C_c(\mathcal{A})$.

6.1 Definition of the crossed product and basic properties

In this section we aim at defining the (*-algebraic) crossed product of $C_c(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ). For that we are going to define some sort of a bundle over G/Γ, where the fiber over each $g\Gamma$ is precisely $C_c(\mathcal{A}/\Gamma^g)$.

Definition 6.1.1. Let $B(\mathcal{A}, G, \Gamma)$ be the vector space of finitely supported functions $f : G/\Gamma \to M(C_c(\mathcal{A}))$ satisfying the following compatibility condition

$$f(\gamma g\Gamma) = \alpha_\gamma(f(g\Gamma)), \tag{6.1}$$

for all $\gamma \in \Gamma$ and $g\Gamma \in G/\Gamma$.

Lemma 6.1.2. *For every $f \in B(\mathcal{A}, G, \Gamma)$ and $g\Gamma \in G/\Gamma$ we have*

$$f(g\Gamma) \in M(C_c(\mathcal{A}))^{\Gamma^g} .$$

Proof : This follows directly from the compatibility condition (6.1), since for every $\gamma \in \Gamma^g$ we have

$$\alpha_\gamma(f(g\Gamma)) = f(\gamma g\Gamma) = f(g\Gamma) .$$

$\square$

Definition 6.1.3. The vector subspace of $B(\mathcal{A}, G, \Gamma)$ consisting of the functions $f : G/\Gamma \to M(C_c(\mathcal{A}))$ satisfying the compatibility condition (6.1) and the property

$$f(g\Gamma) \in C_c(\mathcal{A}/\Gamma^g) , \tag{6.2}$$

will be denoted by $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and will be called the *-*algebraic crossed product* of $C_c(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ).

It is relevant to point out that the definitions of the spaces $B(\mathcal{A}, G, \Gamma)$ and $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ seem more suitable for Hecke pairs (G, Γ), as in general a function in $B(\mathcal{A}, G, \Gamma)$ could only have support on those elements $g\Gamma \in G/\Gamma$ such that $|\Gamma g\Gamma/\Gamma| < \infty$.

We now define a product and an involution in $B(\mathcal{A}, G, \Gamma)$ by:

$$(f_1 * f_2)(g\Gamma) := \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\, \alpha_h(f_2(h^{-1}g\Gamma)) , \tag{6.3}$$

$$(f^*)\,(g\Gamma) := \Delta(g^{-1})\alpha_g(f(g^{-1}\Gamma))^* . \tag{6.4}$$

Proposition 6.1.4. *$B(\mathcal{A}, G, \Gamma)$ becomes a unital *-algebra under the product and involution defined above, whose identity element is the function f such that $f(\Gamma) = 1$ and is zero in the remaining points of G/Γ.*

Proof: First, we claim that the expression for the product defined above is well-defined in $B(\mathcal{A}, G, \Gamma)$, i.e. for $f_1, f_2 \in B(\mathcal{A}, G, \Gamma)$ the expression

$$(f_1 * f_2)(g\Gamma) := \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\, \alpha_h(f_2(h^{-1}g\Gamma))$$

is independent from the choice of the representatives $[h]$ and also that it has finitely many summands. Independence from the choice of the representatives $[h] \in G/\Gamma$ follows directly from the compatibility condition (6.1) and the fact that the sum is finite follows simply from the fact that f_1 has finite support.

Now we claim that $f_1 * f_2$ has also finite support, for $f_1, f_2 \in B(\mathcal{A}, G, \Gamma)$. Let $S_1, S_2 \subseteq G/\Gamma$ be the supports of the functions f_1 and f_2 respectively. We will regard S_1 and S_2 as subsets of G (being finite unions of left cosets). It is easy to check that the function $G \times G \to M(C_c(\mathcal{A}))$

$$(h, g) \mapsto f_1(h\Gamma)\alpha_h(f_2(h^{-1}g\Gamma))$$

has support contained in $S_1 \times (S_1 \cdot S_2)$. Since (G, Γ) is a Hecke pair, the product $S_1 \cdot S_2$ is also a finite union of left cosets. Hence, $f_1 * f_2$ has finite support.

We also notice that $f_1 * f_2$ satisfies the compatibility condition (6.1), thus defining an element of $B(\mathcal{A}, G, \Gamma)$, since for any $\gamma \in \Gamma$ we have

$$\begin{aligned}
(f_1 * f_2)(\gamma g\Gamma) &= \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\, \alpha_h(f_2(h^{-1}\gamma g\Gamma)) \\
&= \sum_{[h] \in G/\Gamma} f_1(\gamma h\Gamma)\, \alpha_{\gamma h}(f_2(h^{-1}g\Gamma)) \\
&= \sum_{[h] \in G/\Gamma} \alpha_\gamma(f_1(h\Gamma))\, \alpha_\gamma \circ \alpha_h(f_2(h^{-1}g\Gamma)) \\
&= \alpha_\gamma\big((f_1 * f_2)(g\Gamma)\big) .
\end{aligned}$$

In a similar way we can see that the expression that defines the involution is well-defined in $B(\mathcal{A}, G, \Gamma)$. There are now a few things that need to be checked before we can say that $B(\mathcal{A}, G, \Gamma)$ is a *-algebra, namely that the product is associative and the involution is indeed an involution relatively to this product (the fact that the product is distributive and the properties concerning multiplication by scalars are obvious). The proofs of these facts are essentially just a mimic of the corresponding proofs for "classical" crossed products by groups. Thus, we can say that $B(\mathcal{A}, G, \Gamma)$ is *-algebra under this product and involution. $\square$

Theorem 6.1.5. *$C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a *-ideal of $B(\mathcal{A}, G, \Gamma)$. In particular it is a *-algebra for the above operations.*

Proof: It is easy to see that the space $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is invariant for the involution, i.e.

$$f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \implies f^* \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma\,.$$

Thus, to prove that $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ is a (two-sided) *-ideal of $B(\mathcal{A}, G, \Gamma)$ it is enough to prove that is a right ideal, i.e. if $f_1 \in B(\mathcal{A}, G, \Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ then $f_1 * f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, because any right *-ideal is automatically two-sided. Hence, all we need to prove is that $(f_1 * f_2)(g\Gamma) \in C_c(\mathcal{A}/\Gamma^g)$, for every $f_1 \in B(\mathcal{A}, G, \Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. The proof of this fact will follow the following steps:

1) Prove that: given a subgroup $H \subseteq G$, $f \in C_c(\mathcal{A}/H)$ and a unit $u \in X^0$, we have $f \cdot 1_u \in C_c(\mathcal{A})$.

2) Let $T := (f_1 * f_2)(g\Gamma) = \sum_{[h]\in G/\Gamma} f_1(h\Gamma)\alpha_h(f_2(h^{-1}g\Gamma))$. Use 1) to show that $T \cdot 1_u \in C_c(\mathcal{A})$ for any unit $u \in X^0$.

3) Fix a unit $u \in X^0$. By 2) we have $T\,1_u = \sum_i (a_i)_{x_i}$, where the elements $x_i \in X$ are such that $\mathbf{s}(x_i) = u$. Show that $T\,1_{u\Gamma^g} = \sum_i (a_i)_{x_i\Gamma^g}$, and conclude that $T\,1_{u\Gamma^g} \in C_c(\mathcal{A}/\Gamma^g)$.

4) Prove that there exists a finite set of units $\{u_1, \dots, u_n\} \subseteq X^0$ such that $T = \sum_{i=1}^n T\,1_{u_i\Gamma^g}$. Conclude that $T \in C_c(\mathcal{A}/\Gamma^g)$.

- Proof of 1) : This follows immediately from Proposition 5.3.5.
- Proof of 2) : We know that $f_2(h^{-1}g\Gamma) \in C_c(\mathcal{A}/\Gamma^{h^{-1}g})$. Thus, from Proposition 5.3.9, we conclude that $\alpha_h(f_2(h^{-1}g)) \in C_c\big(\mathcal{A}/h\Gamma h^{-1} \cap g\Gamma g^{-1}\big)$. Now, using 1), we see that $\alpha_h(f_2(h^{-1}g))\,1_u \in C_c(\mathcal{A})$ and consequently $f_1(h\Gamma)\alpha_h(f_2(h^{-1}g))\,1_u \in C_c(\mathcal{A})$. Hence, $T\,1_u \in C_c(\mathcal{A})$.
- Proof of 3) : For any $\gamma \in \Gamma^g$ we have, using Lemma 6.1.2,

$$\begin{aligned} T\,1_{u\gamma} &= \alpha_{\gamma^{-1}}(T)\,1_{u\gamma} = \alpha_{\gamma^{-1}}\big(T\alpha_\gamma(1_{u\gamma})\big) \\ &= \alpha_{\gamma^{-1}}\big(T\,1_u)\big) = \sum_i (a_i)_{x_i\gamma}\,. \end{aligned}$$

Let $y \in X$ and $b \in \mathcal{A}_y$. We have

$$T\, 1_{u\Gamma^g}\, b_y \;=\; \begin{cases} T b_y\,, & \text{if } \mathbf{r}(y) \in u\Gamma^g \\ 0, & \text{otherwise} \end{cases} .$$

Assume now that $\mathbf{r}(y) \in u\Gamma^g$ and let $\widetilde{\gamma} \in \Gamma^g$ be such that $\mathbf{r}(y) = u\widetilde{\gamma}$. We then have

$$Tb_y \;=\; T\, 1_{u\widetilde{\gamma}}\, b_y \;=\; \sum_i (a_i)_{x_i\widetilde{\gamma}}\, b_y\,.$$

Since $\mathbf{s}(x_i) = u$, we have $\mathbf{s}(x_i\widetilde{\gamma}) = u\widetilde{\gamma} = \mathbf{r}(y)$. Hence,

$$Tb_y \;=\; \sum_i (a_i b)_{x_i\widetilde{\gamma}y}\,.$$

We conclude that

$$\begin{aligned} T\, 1_{u\Gamma^g}\, b_y \;&=\; \begin{cases} \sum_i (a_i b)_{x_i\widetilde{\gamma}y}\,, & \text{if } \mathbf{r}(y) \in u\Gamma^g \\ 0, & \text{otherwise} \end{cases} . \\ &=\; \sum_i (a_i)_{x_i\Gamma^g}\, b_y\,. \end{aligned}$$

Thus, $T\, 1_{u\Gamma^g} = \sum_i (a_i)_{x_i\Gamma^g} \in C_c(\mathcal{A}/\Gamma^g)$.

- Proof of 4) : For easiness of reading of this last part of the proof we introduce the following definition: given $F \in M(C_c(\mathcal{A}))$ we define the *support* of F to be the set $\{u \in X^0 : F\, 1_u \neq 0\}$. Notice in particular that the support of an element a_{xH}, with $a \neq 0$, is the set $\mathbf{s}(x)H$.

  Since $\alpha_h(f_2(h^{-1}g\Gamma)) \in C_c(\mathcal{A}/h\Gamma h^{-1} \cap g\Gamma g^{-1})$, there exists a finite number of units $v_1, \dots, v_k \in X^0$ such that $\alpha_h(f_2(h^{-1}g\Gamma))$ has support in

$$\bigcup_{i=1}^{k} v_i\big(h\Gamma h^{-1} \cap g\Gamma g^{-1}\big) \;\subseteq\; \bigcup_{i=1}^{k} v_i g\Gamma g^{-1}\,.$$

  Hence, there is a finite number of units $w_1, \dots, w_l \in X^0$ such that T has support contained in

$$\bigcup_{i=1}^{l} w_i g\Gamma g^{-1}\,.$$

Therefore, T has support contained in

$$\bigcup_{i=1}^{l}\bigcup_{j=1}^{m} w_i\theta_j\Gamma^g\,,$$

where $\theta_1, \dots, \theta_m$ are representatives of the classes of $g\Gamma g^{-1}/\Gamma^g$ (being a finite number because (G, Γ) is a Hecke pair). Thus, we have proven that there is a finite number of units $u_1, \dots, u_n \in X^0$ such that T has support inside $\bigcup_{i=1}^{n} u_i\Gamma^g$. Moreover, we can suppose we have chosen the units $u_1, \dots, u_n$ such that the corresponding orbits $u_i\Gamma^g$ are mutually disjoint. It is now easy to see that we have $T = \sum_{i=1}^{n} T\, 1_{u_i\Gamma^g}$. Indeed, given $y \in X$ and $b \in \mathcal{A}_y$, if $\mathbf{r}(y) \notin \bigcup_{i=1}^{n} u_i\Gamma^g$, then

$$Tb_y = T\, 1_{\mathbf{r}(y)}\, b_y = 0 = \sum_{i=1}^{n} T\, 1_{u_i\Gamma^g}\, b_y\,,$$

and if $\mathbf{r}(y) \in \bigcup_{i=1}^{n} u_i\Gamma^g$, then $\mathbf{r}(y)$ belongs to precisely one of the orbits, say $u_{i_0}\Gamma^g$, and we have

$$\sum_{i=1}^{n} T\, 1_{u_i\Gamma^g}\, b_y = T 1_{u_{i_0}\Gamma^g}\, b_y = Tb_y\,.$$

Hence, we must have $T = \sum_{i=1}^{n} T\, 1_{u_i\Gamma^g}$, and by 3) we conclude that $T \in C_c(\mathcal{A}/\Gamma^g)$. $\square$

Remark 6.1.6. If Γ is a normal subgroup of G it is easy to see that the action α gives rise to an action of G/Γ on $C_c(\mathcal{A}/\Gamma)$ and in this case $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is nothing but the usual (*-algebraic) crossed product of $C_c(\mathcal{A}/\Gamma)$ by the group G/Γ. In a similar fashion, the algebra $B(\mathcal{A}, G, \Gamma)$ is nothing but the usual (*-algebraic) crossed product of $M(C_c(\mathcal{A}/\Gamma))$ by the group G/Γ. The reason for considering the larger algebra $B(\mathcal{A}, G, \Gamma)$ shall be clarified later in Remark 6.1.13.

As it is well-known, when working with crossed products $A \times G$ by discrete groups, one always has an embedded copy of A inside the crossed product. Something analogous happens in the case of crossed products by Hecke pairs, where $C_c(\mathcal{A}/\Gamma)$ is canonically embedded in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, as is stated in the next result (whose proof amounts to routine verification).

Proposition 6.1.7. *There is a natural embedding of the* **-algebra* $C_c(\mathcal{A}/\Gamma)$ *in* $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$*, which identifies an element* $f \in C_c(\mathcal{A}/\Gamma)$ *with the function* $T_f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *such that*

$$T_f(\Gamma) = f \qquad \textit{and} \qquad T_f \textit{ is zero elsewhere}.$$

Remark 6.1.8. The above result says that we can identify $C_c(\mathcal{A}/\Gamma)$ with the functions of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ with support in Γ. We shall, henceforward, make no distinctions in notation between an element of $C_c(\mathcal{A}/\Gamma)$ and its correspondent in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.

Theorem 6.1.9. $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *is an essential* **-ideal of* $B(\mathcal{A}, G, \Gamma)$*. In particular,* $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *is an essential* **-algebra. Moreover, there are natural embeddings*

$$C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \hookrightarrow B(\mathcal{A}, G, \Gamma) \hookrightarrow M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\,,$$

that make the following diagram commute

$$\begin{tikzcd}
 & M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) \\
C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \arrow[ur, "L"] \arrow[r] & B(\mathcal{A}, G, \Gamma)\,. \arrow[u]
\end{tikzcd}$$

Proof: We have already proven that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a *-ideal of $B(\mathcal{A}, G, \Gamma)$, thus we only need to check that this ideal is in fact essential. Suppose $f \in B(\mathcal{A}, G, \Gamma)$ is such that $f * \big(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma\big) = \{0\}$. Then, in particular, using Proposition 6.1.7, we must have $f * \big(C_c(\mathcal{A}/\Gamma)\big) = \{0\}$. Let $g\Gamma \in G/\Gamma$ and take $a_{xg\Gamma} \in C_c(\mathcal{A}/\Gamma)$, we then have

$$0 = \Big(f * (a_{xg\Gamma})\Big)\,(g\Gamma) = f(g\Gamma)\alpha_g(a_{xg\Gamma}) = f(g\Gamma)a_{xg\Gamma g^{-1}}\,.$$

Thus, multiplying by $1_{\mathbf{s}(x)} \in M(C_c(\mathcal{A}))$ we get

$$0 = f(g\Gamma)a_{xg\Gamma g^{-1}}1_{\mathbf{s}(x)} = f(g\Gamma)a_x\,.$$

Since this true for all $a \in \mathcal{A}_x$ and $x \in X$ we must have $f(g\Gamma) = 0$. Thus, $f = 0$ and we conclude that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is an essential *-ideal of $B(\mathcal{A}, G, \Gamma)$.

Since $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a *-subalgebra of $B(\mathcal{A}, G, \Gamma)$, we immediately conclude that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is an essential *-algebra.

The embedding of $B(\mathcal{A}, G, \Gamma)$ in $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$ then follows from the universal property of multiplier algebras, Theorem 4.2.11. $\square$

In the theory of crossed products $A \times G$ by groups, one always has an embedded copy of the group algebra $\mathbb{C}(G)$ inside the multiplier algebra $M(A \times G)$. Something analogous happens in the case of crossed products by Hecke pairs, where the Hecke algebra $\mathcal{H}(G, \Gamma)$ is canonically embedded in the multiplier algebra $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$, as is stated in the next result (whose proof amounts to routine verification).

Proposition 6.1.10. *The Hecke *-algebra $\mathcal{H}(G, \Gamma)$ embeds in $B(\mathcal{A}, G, \Gamma)$ in the following way: an element $f \in \mathcal{H}(G, \Gamma)$ is identified with the element $\widetilde{f} \in B(\mathcal{A}, G, \Gamma)$ given by*

$$\widetilde{f}(g\Gamma) := f(\Gamma g \Gamma)\mathbf{1}\,,$$

where $\mathbf{1}$ is the unit of $M(C_c(\mathcal{A}))$.

The next result does not typically play an essential role in the case of crossed products by groups, but will be extremely important for us in case of crossed products by Hecke pairs. The proof is also just routine verification.

Proposition 6.1.11. *The algebra $C_c(X^0/\Gamma)$ embeds in $B(\mathcal{A}, G, \Gamma)$ in the following way: an element $f \in C_c(X^0/\Gamma)$ is identified with the function $T_f \in B(\mathcal{A}, G, \Gamma)$ given by*

$$T_f(\Gamma) = f \qquad \text{and} \qquad T_f \ \text{is zero elsewhere}\,.$$

Remark 6.1.12. Propositions 6.1.10 and 6.1.11 allow us to view both the Hecke *-algebra $\mathcal{H}(G, \Gamma)$ and $C_c(X^0/\Gamma)$ as *-subalgebras of $B(\mathcal{A}, G, \Gamma)$. We shall henceforward make no distinctions in notation between an element of $\mathcal{H}(G, \Gamma)$ or $C_c(X^0/\Gamma)$ and its correspondent in $B(\mathcal{A}, G, \Gamma)$.

The purpose of the following diagram is to illustrate, in a more condensed form, all the canonical embeddings we have been considering so far:

$$
\begin{tikzcd}
C_c(\mathcal{A}/\Gamma) \arrow[r] & C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \arrow[dr] & & \\
\mathcal{H}(G,\Gamma) \arrow[rr] & & B(\mathcal{A},G,\Gamma) \arrow[r] & M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)) \\
C_c(X^0/\Gamma) \arrow[urr] & & &
\end{tikzcd}
$$

Remark 6.1.13. The reason for considering the algebra $B(\mathcal{A}, G, \Gamma)$ is two-fold. On one side $B(\mathcal{A}, G, \Gamma)$ made it easier to make sure the convolution product (6.3) was well-defined in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. On the other (perhaps more important) side, the fact that both $\mathcal{H}(G, \Gamma)$ and $C_c(X^0/\Gamma)$ are canonically embedded in $B(\mathcal{A}, G, \Gamma)$ allows us to treat the elements of $\mathcal{H}(G, \Gamma)$ and $C_c(X^0/\Gamma)$ both as multipliers in $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma))$, but also allows us to operate these elements with the convolution product and involution expressions (6.3) and (6.4), as these are defined in $B(\mathcal{A}, G, \Gamma)$.

As it is well-known in the theory of crossed products by discrete groups, a (*-algebraic) crossed product $A \times G$ is spanned by elements of the form $a * g$, where $a \in A$ and $g \in G$ (here g is seen as an element of the group algebra $\mathbb{C}(G) \subseteq M(A \times G)$). We will now explore something analogous in the case of crossed products by Hecke pairs. It turns out that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is spanned by elements of the form $a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g\Gamma}$, where $x \in X$, $a \in \mathcal{A}_x$ and $g\Gamma \in G/\Gamma$, as we show in the next result.

Theorem 6.1.14. *For any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have*

$$
f = \sum_{[g] \in \Gamma \backslash G/\Gamma} \; \sum_{x\Gamma^g \in X/\Gamma^g} \Big(f(g\Gamma)(x\Gamma^g) \Big)_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g\Gamma} \,. \tag{6.5}
$$

In particular, $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is spanned by elements of the form

$$
a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g\Gamma} \,,
$$

with $x \in X$, $a \in \mathcal{A}_x$ and $g\Gamma \in G/\Gamma$.

The following lemma is needed in order to prove the above result:

Lemma 6.1.15. *Let $x \in X$, $a \in \mathcal{A}_x$ and $g\Gamma \in G/\Gamma$. We have*

$$a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\ (h\Gamma) = \begin{cases} a_{x\gamma^{-1}\Gamma^{\gamma g}}\,, & \textit{if } h\Gamma = \gamma g\Gamma, \ \textit{with } \gamma \in \Gamma \\ 0, & \textit{otherwise} \ . \end{cases}$$

In particular,

$$a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\ (g\Gamma) = a_{x\Gamma^g}\,.$$

Proof: An easy computation yields

$$a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\ (h\Gamma) \ = \ a_{x\Gamma} \cdot \Gamma g\Gamma(h\Gamma) \cdot \alpha_h(1_{\mathbf{s}(x)g\Gamma})\,,$$

from which we conclude that $a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ is supported in the double coset $\Gamma g\Gamma$. Now, evaluating at the point $g\Gamma \in G/\Gamma$ we get

$$\begin{aligned} a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\ (g\Gamma) &= a_{x\Gamma} \cdot \Gamma g\Gamma(g\Gamma) \cdot \alpha_g(1_{\mathbf{s}(x)g\Gamma}) \\ &= a_{x\Gamma} \cdot \alpha_g(1_{\mathbf{s}(x)g\Gamma}) \\ &= a_{x\Gamma} \cdot 1_{\mathbf{s}(x)g\Gamma g^{-1}} \\ &= a_{x\Gamma^g}\,, \end{aligned}$$

where the last equality comes from Proposition 5.3.10. From the compatibility condition (6.1) and Proposition 5.3.9 it then follows that, for $\gamma \in \Gamma$,

$$\begin{aligned} a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\ (\gamma g\Gamma) &= \alpha_\gamma(a_{x\Gamma^g}) \\ &= a_{x\gamma^{-1}\Gamma^{\gamma g}}\,. \end{aligned}$$

$\square$

Proof of Theorem 6.1.14: Let us first prove that the expression on the right hand side of (6.5) is well-defined. It is easy to see that for every $g \in G$, the expression

$$\sum_{x\Gamma^g \in X/\Gamma^g} \Big(f(g\Gamma)(x\Gamma^g)\Big)_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$$

does not depend on the choice of the representative x of $x\Gamma^g$. Now, let us see that it also does not depend on the choice of the representative g in $\Gamma g\Gamma$. Let $\gamma g\theta$, with $\gamma, \theta \in \Gamma$, be any other representative. We have

$$\begin{aligned}
&\sum_{x\Gamma^{\gamma g\theta} \in X/\Gamma^{\gamma g\theta}} \Big(f(\gamma g\theta\Gamma)(x\Gamma^{\gamma g\theta})\Big)_{x\Gamma} * \Gamma\gamma g\theta\Gamma * 1_{\mathbf{s}(x)\gamma g\theta\Gamma} = \\
&= \sum_{x\Gamma^{\gamma g} \in X/\Gamma^{\gamma g}} \Big(f(\gamma g\Gamma)(x\Gamma^{\gamma g})\Big)_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma} \\
&= \sum_{x\Gamma^{\gamma g} \in X/\Gamma^{\gamma g}} \Big(\alpha_\gamma(f(g\Gamma))(x\Gamma^{\gamma g})\Big)_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma} \\
&= \sum_{x\Gamma^{\gamma g} \in X/\Gamma^{\gamma g}} \Big((f(g\Gamma)(x\gamma\Gamma^{g})\Big)_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}
\end{aligned}$$

We notice that there is a well-defined bijective correspondence $X/\Gamma^g \to X/\Gamma^{\gamma g}$ given by $x\Gamma^g \mapsto x\gamma^{-1}\Gamma^{\gamma g}$. Thus, we get

$$\begin{aligned}
&= \sum_{x\Gamma^{g} \in X/\Gamma^{g}} \Big(f(g\Gamma)(x\gamma^{-1}\gamma\Gamma^{g})\Big)_{x\gamma^{-1}\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x\gamma^{-1})\gamma g\Gamma} \\
&= \sum_{x\Gamma^{g} \in X/\Gamma^{g}} \Big(f(g\Gamma)(x\Gamma^{g})\Big)_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\,.
\end{aligned}$$

Hence, the expression in (6.5) is well-defined. Let us now prove the decomposition in question. For any $t\Gamma \in G/\Gamma$ we have

$$\begin{aligned}
&\sum_{[g]\in\Gamma\backslash G/\Gamma}\ \sum_{x\Gamma^{g} \in X/\Gamma^{g}} \Big(f(g\Gamma)(x\Gamma^{g})\Big)_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\ (t\Gamma) = \\
&= \sum_{x\Gamma^{t} \in X/\Gamma^{t}} \Big(f(t\Gamma)(x\Gamma^{t})\Big)_{x\Gamma} * \Gamma t\Gamma * 1_{\mathbf{s}(x)t\Gamma}\ (t\Gamma)\,.
\end{aligned}$$

By Lemma 6.1.15 it follows that

$$\begin{aligned}
&= \sum_{x\Gamma^{t} \in X/\Gamma^{t}} \Big(f(t\Gamma)(x\Gamma^{t})\Big)_{x\Gamma^t} \\
&= f(t\Gamma)\,,
\end{aligned}$$

and this finishes the proof. $\square$

In the following result we collect some useful equalities concerning products in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, which will be useful later on. One should observe the similarities between the equalities (6.8) and (6.9) and the equalities obtained by an Huef, Kaliszewski and Raeburn in [19, Lemma 1.3 (i) and (ii)] if in their setting one was allowed to somehow "drop" the representations. The similarity is more than a coincidence and we will address this later in Chapter 10.

Proposition 6.1.16. *In $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ the following equalities hold:*

$$\big(a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^* = \Delta(g)\ a^*_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1})\Gamma}\,, \tag{6.6}$$

$$1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * a_{xg\Gamma} = a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}\,, \tag{6.7}$$

$$a_{x\Gamma} * \Gamma g \Gamma = \sum_{[\gamma]\in \mathcal{S}_x\backslash\Gamma/\Gamma^g} a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\,. \tag{6.8}$$

$$\Gamma g \Gamma * a_{x\Gamma} = \sum_{[\gamma]\in \mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\gamma g^{-1}\Gamma} * \Gamma g \Gamma * a_{x\Gamma}\,. \tag{6.9}$$

In particular, from (6.7) we see that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is also spanned by all elements of the form $1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{xg\Gamma}$, with $g \in G$, $x \in X$ and $a \in \mathcal{A}_x$.

Proof: Let us first prove equality (6.6). First we notice that

$$\big(a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^* = \Delta(g)\ 1_{\mathbf{s}(x)g\Gamma} * \Gamma g^{-1}\Gamma * a^*_{x^{-1}\Gamma}\,,$$

which means that $\big(a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^*$ has support in the double coset $\Gamma g^{-1}\Gamma$. Now evaluating this element on $g^{-1}\Gamma$ we get,

$$\begin{aligned}
\big(a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^*(g^{-1}\Gamma) &= \Delta(g)\ \alpha_{g^{-1}}\big(\big(a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)(g\Gamma)\big)^* \\
&= \Delta(g)\ \alpha_{g^{-1}}(a_{x\Gamma^g})^* \\
&= \Delta(g)\ a^*_{x^{-1}g\Gamma g^{-1}} \\
&= \Delta(g)\ \big(a^*_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1})\Gamma}\big)(g^{-1}\Gamma)\,.
\end{aligned}$$

Let us now prove equality (6.7). We have

$$\begin{aligned}
1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * a_{xg\Gamma} &= \Delta(g)\ \big(a^*_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma}\big)^* \\
&= \Delta(g)\ \big(a^*_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1}g)g^{-1}\Gamma}\big)^*,
\end{aligned}$$

which together with (6.6) yields

$$\begin{aligned} &= \Delta(g)\Delta(g^{-1})\, a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(xg)\Gamma} \\ &= a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(xg)\Gamma}\,. \end{aligned}$$

Let us now prove (6.8). An easy computation yields

$$a_{x\Gamma} * \Gamma g\Gamma\ (h\Gamma) \;=\; a_{x\Gamma} \cdot \Gamma g\Gamma(h\Gamma)\,,$$

from which we conclude that $a_{x\Gamma} * \Gamma g\Gamma$ has support in $\Gamma g\Gamma$. Evaluating this element on the point $g\Gamma$ we get

$$a_{x\Gamma} * \Gamma g\Gamma\ (g\Gamma) \;=\; a_{x\Gamma} \cdot \Gamma g\Gamma(g\Gamma) \;=\; a_{x\Gamma}\,.$$

From Proposition 5.3.7 one always has the following decomposition

$$a_{x\Gamma} \;=\; \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} a_{x\gamma\Gamma^g}\,.$$

Together with Proposition 6.1.15 we get

$$\begin{aligned} a_{x\Gamma} * \Gamma g\Gamma\ (g\Gamma) &= a_{x\Gamma} \\ &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} a_{x\gamma\Gamma^g} \\ &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} a_{x\gamma\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\ (g\Gamma) \\ &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\ (g\Gamma)\,, \end{aligned}$$

and equality (6.8) is proven.

Equality (6.9) follows easily from (6.8) by taking the involution and using the fact that $\mathcal{S}_x = \mathcal{S}_{x^{-1}}$.

The last claim of this proposition follows simply from (6.7) and Proposition 6.1.14. $\square$

In the theory of crossed products $A \times G$ by discrete groups one has a "covariance relation" of the form $g * a * g^{-1} = \alpha_g(a)$. This relation is essential in the passage from covariant representations of the system (A, G, α) to representations of the crossed product. More generally, the following relation holds in $A \times G$:

$$g * a * h \;=\; \alpha_g(a) * gh\,.$$

We will now explore how this generalizes to the setting of crossed products by Hecke pairs. What we are aiming for is a description of how products of the form $\Gamma g \Gamma * a_{x\Gamma} * \Gamma s \Gamma$ can be expressed by the canonical spanning set of elements of the form $b_{y\Gamma} * \Gamma h \Gamma * 1_{\mathbf{s}(x)h\Gamma}$ (according to Theorem 6.1.14). This will be achieved in Corollary 6.1.19 below and will play an important role in the representation theory of crossed products by Hecke pairs, particularly in the definition of covariant representations. One should observe the similarities between the expressions we obtain both in Theorem 6.1.17 and Corollary 6.1.19 and the expression provided by an Huef, Kaliszewski and Raeburn in [19, Definition 1.1] (if one "forgets" the representations in their setting). Once again, this is more than a coincidence as we will see in Chapter 10. In fact, an Huef, Kaliszewski and Raeburn's definition served as a guiding line for our results below and for the definition of a covariant representation (Definition 6.2.1) which we shall present in the next section.

Before we establish the results we are aiming for we need to establish some notation, which will be used throughout this work. For $w, v \in G$ and a unit $y \in X^0$ we define the sets

$$\mathfrak{n}^y_{w,v} := \left\{ [r] \in \Gamma w \Gamma/\Gamma : r^{-1} w v \Gamma \subseteq \Gamma v \Gamma \text{ and } \; y w^{-1} \in y \Gamma r^{-1} \right\}, \tag{6.10}$$

$$\mathfrak{d}^y_{w,v} := \left\{ [r] \in \Gamma w \Gamma/\Gamma : r^{-1} w v \Gamma \subseteq \Gamma v \Gamma \text{ and } \; y w^{-1} \in y \Gamma r^{-1} \Gamma^{wv} \right\}. \tag{6.11}$$

and the numbers

$$n^y_{w,v} := \#\, \mathfrak{n}^y_{w,v}\,, \tag{6.12}$$

$$d^y_{w,v} := \#\, \mathfrak{d}^y_{w,v}\,, \tag{6.13}$$

$$N^y_{w,v} := \frac{n^y_{w,v}}{d^y_{w,v}}\,. \tag{6.14}$$

We will also denote by $E^y_{u,v}$ the double coset space

$$E^y_{u,v} := \mathcal{S}_y \backslash \Gamma / (u \Gamma u^{-1} \cap v \Gamma v^{-1})\,. \tag{6.15}$$

Theorem 6.1.17. *Let $g, s \in G$ and $y \in X^0$. We have that*

$$\begin{aligned}
\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma &= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right) \\
&= \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{L(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right) \\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g)N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right).
\end{aligned}$$

In order to prove the above result we will need the following lemma, which gives some properties of the numbers $n^y_{w,v}$ and $d^y_{w,v}$.

Lemma 6.1.18. *Let $w, v, \in G$, $\theta \in \Gamma$ and $y \in X^0$. The numbers $n^y_{w,v}$ and $d^y_{w,v}$ satisfy the following properties:*

i) $n^y_{w,v\theta} = n^y_{w,v}$ *i')* $d^y_{w,v\theta} = d^y_{w,v}$

ii) $n^y_{\theta w,v} = n^y_{w,v}$ *ii')* $d^y_{\theta w,v} = d^y_{w,v}$

iii) $n^{y\theta}_{w,\theta^{-1}v} = n^y_{w\theta^{-1},v}$ *iii')* $d^{y\theta}_{w,\theta^{-1}v} = d^y_{w\theta^{-1},v}$

More generally, if $\widetilde{w}, \widetilde{v} \in G$ and $\widetilde{y} \in X^0$ are such that $\Gamma\widetilde{w}\Gamma = \Gamma w\Gamma$, $\Gamma\widetilde{v}\Gamma = \Gamma v\Gamma$, $\widetilde{y}\Gamma = y\Gamma$, $\widetilde{w}\widetilde{v}\Gamma = wv\Gamma$ and $\widetilde{y}\widetilde{w}^{-1}\Gamma^{wv} = yw^{-1}\Gamma^{wv}$, then

iv) $n^y_{w,v} = n^{\widetilde{y}}_{\widetilde{w},\widetilde{v}}$ *iv')* $d^y_{w,v} = d^{\widetilde{y}}_{\widetilde{w},\widetilde{v}}$

Proof: Assertions $i)$ and $i')$ are obvious.

Assertion $ii)$ follows from the observation that $[r] \mapsto [\theta^{-1}r]$ establishes a bijection between the sets $\mathfrak{n}^y_{w,v}$ and $\mathfrak{n}^y_{\theta w,v}$.

Assertion $ii')$ is proven in a similar fashion as assertion $ii)$.

To prove assertion $iv)$, let $\theta \in \Gamma^{wv}$ be such that $\widetilde{y}\widetilde{w}^{-1} = yw^{-1}\theta$. We have

$$\begin{aligned}
\mathfrak{n}^{\widetilde{y}}_{\widetilde{w},\widetilde{v}} &= \left\{[r] \in \Gamma\widetilde{w}\Gamma/\Gamma : r^{-1}\widetilde{w}\widetilde{v}\Gamma \subseteq \Gamma\widetilde{v}\Gamma \text{ and } \widetilde{y}\widetilde{w}^{-1} \in \widetilde{y}\Gamma r^{-1}\right\} \\
&= \left\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } yw^{-1}\theta \in y\Gamma r^{-1}\right\}.
\end{aligned}$$

Since $\theta \in \Gamma^{wv}$ we have $\theta wv\Gamma = wv\Gamma$, so that

$$\begin{aligned}
&= \left\{[r] \in \Gamma\theta^{-1}w\Gamma/\Gamma : r^{-1}\theta^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } yw^{-1}\theta \in y\Gamma r^{-1}\right\} \\
&= \mathfrak{n}^y_{\theta^{-1}w,v}.
\end{aligned}$$

Now, from assertion ii), it follows that $n^{\widetilde{y}}_{\widetilde{w},\widetilde{v}} = n^{y}_{\theta^{-1}w,v} = n^{y}_{w,v}$.

As for assertion iv'), taking $\theta \in \Gamma^{wv}$ again as such that $\widetilde{y}\widetilde{w}^{-1} = yw^{-1}\theta$, we notice that

$$\begin{aligned}
\mathfrak{d}^{\widetilde{y}}_{\widetilde{w},\widetilde{v}} &= \big\{[r] \in \Gamma\widetilde{w}\Gamma/\Gamma : r^{-1}\widetilde{w}\widetilde{v}\Gamma \subseteq \Gamma\widetilde{v}\Gamma \text{ and } \ \widetilde{y}\widetilde{w}^{-1} \in \widetilde{y}\Gamma r^{-1}\Gamma^{\widetilde{w}\widetilde{v}}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ yw^{-1}\theta \in y\Gamma r^{-1}\Gamma^{wv}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ yw^{-1} \in y\Gamma r^{-1}\Gamma^{wv}\big\} \\
&= \mathfrak{d}^{y}_{w,v}\,.
\end{aligned}$$

Assertions iii) and iii') are a direct consequence of iv) and iv'). $\square$

Proof of Theorem 6.1.17: We have

$$\begin{aligned}
\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma\ (t\Gamma) &= \sum_{[w]\in G/\Gamma} \Gamma g\Gamma(w\Gamma)\,\alpha_w\big((1_{y\Gamma} * \Gamma s\Gamma)\ (w^{-1}t\Gamma)\big) \\
&= \sum_{[w]\in \Gamma g\Gamma/\Gamma} \alpha_w\big((1_{y\Gamma} * \Gamma s\Gamma)\ (w^{-1}t\Gamma)\big) \\
&= \sum_{[w]\in \Gamma g\Gamma/\Gamma} \alpha_w\big(1_{y\Gamma} \cdot \Gamma s\Gamma(w^{-1}t\Gamma)\big) \\
&= \sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \alpha_w(1_{y\Gamma}) \\
&= \sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}}
\end{aligned}$$

We now claim that

$$\sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}} \;=\; \sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \ \sum_{[\gamma]\in E^{y}_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^{t}}\,. \tag{6.16}$$

To see this, we will evaluate both the right and left expressions above on all points $x \in X^0$ and see that we obtain the same value. First, we note that if $x \in X^0$ is not of the form $y\theta\widetilde{w}^{-1}$, for some $\theta \in \Gamma$ and $\widetilde{w} \in \Gamma g\Gamma$ such that $\widetilde{w}^{-1}t\Gamma \subseteq \Gamma s\Gamma$, then both expressions are zero. Suppose now that $x = y\theta\widetilde{w}^{-1}$ for some $\widetilde{w} \in \Gamma g\Gamma$ such that $\widetilde{w}^{-1}t\Gamma \subseteq \Gamma s\Gamma$. Evaluating the left expression we get

$$\sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}}(y\theta\widetilde{w}^{-1}) = \sum_{\substack{[w]\in \Gamma \widetilde{w}\Gamma/\Gamma \\ w^{-1}\widetilde{w}\widetilde{w}^{-1}t\Gamma\subseteq\Gamma \widetilde{w}^{-1}t\Gamma}} 1_{y\Gamma w^{-1}}(y\theta\widetilde{w}^{-1}) = n^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\,.$$

As for the right expression, first we observe that if $y\theta\widetilde{w}^{-1} \in y\gamma w^{-1}\Gamma^t$, then by Lemma 6.1.18 *iv*) and *iv'*) we have $N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} = N^{y\gamma}_{w,w^{-1}t}$. Thus, evaluating the right expression we get

$$\begin{aligned}
&\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) = \\
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) \\
&= N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1})
\end{aligned}$$

Using Proposition 1.2.2 we notice that

$$\begin{aligned}
\sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma w^{-1}\Gamma^t} &= \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma(w^{-1}\Gamma w\cap w^{-1}t\Gamma t^{-1}w)w^{-1}} \\
&= 1_{y\Gamma w^{-1}\Gamma^t}\,,
\end{aligned}$$

from which we obtain that,

$$\begin{aligned}
&N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) = \\
&= N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) \\
&= N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma \widetilde{w}\Gamma/\Gamma \\ w^{-1}\widetilde{w}\widetilde{w}^{-1}t\Gamma\subseteq\Gamma\widetilde{w}^{-1}t\Gamma}} 1_{y\Gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) \\
&= N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\; d^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \\
&= n^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\,.
\end{aligned}$$

So, equality (6.16) is established.

Now, by Proposition 6.1.15, we see that

$$\begin{aligned}
&\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^t} = \\
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma t\Gamma * 1_{y\gamma w^{-1}t\Gamma}\big)(t\Gamma)
\end{aligned}$$

Now, using the fact that condition $w^{-1}t\Gamma \subseteq \Gamma s\Gamma$ means that there exists a (necessarily unique) element $[v] \in \Gamma s\Gamma/\Gamma$ such that $w^{-1}t\Gamma = v\Gamma$, or equivalently, $t\Gamma = wv\Gamma$, we obtain

$$= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma t\Gamma * 1_{y\gamma w^{-1}t\Gamma}\right)(t\Gamma)$$

$$= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} N^{y\gamma}_{w,v} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)\,.$$

We now claim that

$$\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} N^{y\gamma}_{w,v} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma) =$$

$$= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

To prove this we note that, given any $[w] \in \Gamma g\Gamma/\Gamma$ and $[v] \in \Gamma s\Gamma/\Gamma$, the element $\left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$ is nonzero if and only if $\Gamma t\Gamma = \Gamma wv\Gamma$, so that we can write

$$\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma) =$$

$$= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma\subseteq\Gamma t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

$$= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=\theta t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

$$= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ \theta wv\Gamma=\theta t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1}\theta^{-1},v}} \frac{N^{y\gamma}_{\theta w,v}}{L(\theta wv)} \left(1_{y\gamma w^{-1}\theta^{-1}\Gamma} * \Gamma \theta wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

$$= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{\theta w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

By Lemma 6.1.18 $ii)$ and $ii')$ we know that $N^{y\gamma}_{\theta w,v} = N^{y\gamma}_{w,v}$, hence

$$
\begin{aligned}
&= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma\\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)\\
&= L(t) \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma\\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)\\
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma\\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} N^{y\gamma}_{w,v} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)\,.
\end{aligned}
$$

Hence, we have proven that

$$
\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right).
$$

Also,

$$
\begin{aligned}
&\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)\\
&= \sum_{\substack{[\theta]\in\Gamma/\Gamma^g\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{g^{-1}\theta^{-1},v}} \frac{N^{y\gamma}_{\theta g,v}}{L(\theta gv)} \left(1_{y\gamma g^{-1}\theta^{-1}\Gamma} * \Gamma \theta gv\Gamma * 1_{y\gamma v\Gamma}\right)\\
&= \sum_{\substack{[\theta]\in\Gamma/\Gamma^g\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)\\
&= \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{L(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right).
\end{aligned}
$$

Moreover, we also have

$$
\begin{aligned}
&\sum_{[v]\in\Gamma s\Gamma/\Gamma}\ \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{L(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)\\
=\ & L(g^{-1}) \sum_{[v]\in\Gamma s\Gamma/\Gamma}\ \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{\Delta(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)\\
=\ & \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}}\\ [v]\in\Gamma s\Gamma/\Gamma}}\ \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{\Delta(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)\\
=\ & \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}}\\ [v]\in\Gamma s\Gamma/\Gamma}}\ \sum_{[\gamma]\in E^y_{g^{-1},\theta^{-1}v}} \frac{\Delta(g)N^{y\gamma}_{g,\theta^{-1}v}}{L(g\theta^{-1}v)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma g\theta^{-1}v\Gamma * 1_{y\gamma \theta^{-1}v\Gamma}\right),
\end{aligned}
$$

but since there is a well-defined bijection $E^y_{\theta g^{-1},v} \to E^y_{g^{-1},\theta^{-1}v}$ given by $[\gamma]\mapsto[\gamma\theta]$, we obtain

$$
= \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}}\\ [v]\in\Gamma s\Gamma/\Gamma}}\ \sum_{[\gamma]\in E^y_{\theta g^{-1},v}} \frac{\Delta(g)N^{y\gamma\theta}_{g,\theta^{-1}v}}{L(g\theta^{-1}v)} \left(1_{y\gamma\theta g^{-1}\Gamma} * \Gamma g\theta^{-1}v\Gamma * 1_{y\gamma\theta\theta^{-1}v\Gamma}\right)
$$

and from Lemma 6.1.18 we get $N^{y\gamma\theta}_{g,\theta^{-1}v} = N^{y\gamma}_{g\theta^{-1},v}$, thus

$$
\begin{aligned}
=\ & \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}}\\ [v]\in\Gamma s\Gamma/\Gamma}}\ \sum_{[\gamma]\in E^y_{\theta g^{-1},v}} \frac{\Delta(g)N^{y\gamma}_{g\theta^{-1},v}}{L(g\theta^{-1}v)} \left(1_{y\gamma\theta g^{-1}\Gamma} * \Gamma g\theta^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right)\\
=\ & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}}\ \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g)N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right).
\end{aligned}
$$

□

Corollary 6.1.19. *Similarly, for $a \in \mathcal{A}_x$ with $x \in X$, we have*

$$\begin{aligned}
\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma &= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)} \left(a_{x\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\right) \\
&= \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},v}} \frac{L(g)N^{\mathbf{s}(x)\gamma}_{g,v}}{L(gv)} \left(a_{x\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\right) \\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(a_{x\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\right).
\end{aligned}$$

Proof: According to equality (6.9) in Proposition 6.1.16 we have

$$\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma = \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\theta g^{-1}\Gamma} * \Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma\,,$$

and by (6.7) in the same proposition we get

$$= \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} a_{x\theta g^{-1}\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma} * \Gamma s\Gamma\,,$$

and by Theorem 6.1.17 we obtain

$$\begin{aligned}
&= \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, a_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma} \\
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}} \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, a_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,.
\end{aligned}$$

For each fixed w, v and γ all the summands in the expression

$$\sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, a_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,,$$

are zero except precisely for one summand and we have

$$\begin{aligned}
&\sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, a_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma} \\
=\ & \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, a_{x\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,.
\end{aligned}$$

Hence we obtain

$$\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma \quad = \quad \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \; \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, a_{x\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,.$$

The remaining equalities in the statement of this corollary are proven in a similar fashion. □

Example 6.1.20. We will now explain how the Hecke algebra $\mathcal{H}(G,\Gamma)$ is an example of a crossed product by a Hecke pair, namely $\mathcal{H}(G,\Gamma) \cong \mathbb{C}\times^{alg} G/\Gamma$, just like group algebras are examples of crossed products by groups.

We start with a groupoid X consisting of only one element, i.e. $X = \{*\}$, and we take the trivial G-action on X. Since the G-action fixes the point $*$ it is indeed H-good and in this case we have $X/\Gamma = X = \{*\}$. We now take the Fell bundle $\mathcal{A}$ over X such that $\mathcal{A}_* = \mathbb{C}$, which trivially has G-invariant fibers. In this case one obviously has that

$$C_c(\mathcal{A}/\Gamma) \cong C_c(X/\Gamma) \cong C_c(X) \cong \mathbb{C}\,.$$

Hence, we are in the conditions of the Standing Assumption 6.0.11 and we can form the crossed product $C_c(\mathcal{A}/\Gamma) \times^{alg}_{\alpha} G/\Gamma$, which we will simply write as $\mathbb{C} \times^{alg}_{\alpha} G/\Gamma$.

Since $\mathbb{C}$ is unital the definitions of $B(\mathcal{A}, G, \Gamma)$ and $\mathbb{C} \times^{alg}_{\alpha} G/\Gamma$ coincide in this case. Moreover Definition 6.1.3 reads that $\mathbb{C} \times^{alg}_{\alpha} G/\Gamma$ is the set of functions $f : G/\Gamma \to \mathbb{C}$ satisfying the compatibility condition (6.1). Since the action α is trivial, the compatibility condition simply says that $\mathbb{C} \times^{alg}_{\alpha} G/\Gamma$ consists of all the functions $f : G/\Gamma \to \mathbb{C}$ which are left Γ-invariant. Morever, the product and involution expressions become respectively

$$(f_1 * f_2)(g\Gamma) := \sum_{[h]\in G/\Gamma} f_1(h\Gamma)\, f_2(h^{-1}g\Gamma)\,,$$
$$(f^*)\,(g\Gamma) := \Delta(g^{-1})\,\overline{f(g^{-1}\Gamma)}\,.$$

Hence, it is clear that $\mathbb{C} \times^{alg}_{\alpha} G/\Gamma$ is nothing but the Hecke algebra $\mathcal{H}(G,\Gamma)$.

It follows from this that the product $\Gamma g\Gamma * 1_{*\Gamma} * \Gamma s\Gamma$ is just the product of the double cosets $\Gamma g\Gamma$ and $\Gamma s\Gamma$ inside the Hecke algebra, since $1_{*\Gamma}$ is the identity element. It is interesting to note in this regard that the expression for this product described in Theorem 6.1.17 is a familiar expression for the product $\Gamma g\Gamma * \Gamma s\Gamma$ in $\mathcal{H}(G,\Gamma)$. To see this, we note that the stabilizer $\mathcal{S}_*$ of $*$ is the whole group G, and therefore $E^*_{u,v}$ consists only of the class $[e]$.

Moreover, the numbers $n^*_{u^{-1},v}$ and $d^*_{u^{-1},v}$, defined in (6.12) and (6.13), are equal, so that $N^*_{u^{-1},v} = 1$. Thus, the expression described in Theorem 6.1.17 is just the usual expression

$$\Gamma g\Gamma * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)}\, \Gamma u^{-1}v\Gamma\,.$$

Example 6.1.21. As a generalization of Example 6.1.20 we will now show that if the G-action fixes every point of the groupoid X, then $C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma$ is isomorphic to the *-algebraic tensor product of $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G,\Gamma)$. This result also has a known analogue in the theory of crossed products by groups.

Proposition 6.1.22. *If the G-action fixes every point of X, then we have*

$$C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma \quad\cong\quad C_c(\mathcal{A}/\Gamma)\odot\mathcal{H}(G,\Gamma)\,,$$

where $\odot$ is the symbol that denotes the *-algebraic tensor product.

Proof: Given that we have canonical embeddings of $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G,\Gamma)$ into $M(C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma)$ we have a canonical linear map from $C_c(\mathcal{A}/\Gamma)\odot\mathcal{H}(G,\Gamma)$ to $M(C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma)$ determined by

$$f_1\otimes f_2 \mapsto f_1 * f_2\,, \tag{6.17}$$

where $f_1\in C_c(\mathcal{A}/\Gamma)$ and $f_2\in\mathcal{H}(G,\Gamma)$. Standard arguments can be used to show that this mapping is injective (since the mappings from both $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G,\Gamma)$ into the multiplier algebra of the crossed product are injections). It is also clear that the image of the map determined by (6.17) is contained in $C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma$. Let us now check that this mapping is surjective. First we will show that the elements of $C_c(\mathcal{A}/\Gamma)$ commute with elements of $\mathcal{H}(G,\Gamma)$ inside $M(C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma)$. It follows from expressions (6.8) and (6.7) that

$$\begin{aligned} a_{x\Gamma} * \Gamma g\Gamma &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma} \\ &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{x\gamma g\Gamma}\,. \end{aligned}$$

Since every point of X is fixed by G we have that $\mathcal{S}_x = G$, and therefore $\mathcal{S}_x \backslash \Gamma / \Gamma^g$ consists only of the class $[e]$, so that we can write

$$= \quad 1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * a_{xg\Gamma} \,.$$

Moreover, since every point of X is fixed by G we can write

$$= \quad 1_{\mathbf{r}(x)g^{-1}\Gamma} * \Gamma g \Gamma * a_{x\Gamma} \,.$$

Now, by the same reasoning as above and using expression (6.9) we have

$$\begin{aligned} &= \sum_{[\gamma] \in \mathcal{S}_x \backslash \Gamma / \Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\gamma g^{-1}\Gamma} * \Gamma g \Gamma * a_{x\Gamma} \\ &= \Gamma g \Gamma * a_{x\Gamma} \,. \end{aligned}$$

Thus we conclude that $a_{x\Gamma} * \Gamma g \Gamma = \Gamma g \Gamma * a_{x\Gamma}$. By Theorem 6.1.14 we know that elements of the form $a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$ span $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, and from commutation relation we just proved it follows that

$$\begin{aligned} a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} &= \Gamma g \Gamma * a_{x\Gamma} * 1_{\mathbf{s}(x)g\Gamma} \\ &= \Gamma g \Gamma * a_{x\Gamma} * 1_{\mathbf{s}(x)\Gamma} \\ &= \Gamma g \Gamma * a_{x\Gamma} \\ &= a_{x\Gamma} * \Gamma g \Gamma \,, \end{aligned}$$

so that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is spanned by elements of the form $a_{x\Gamma} * \Gamma g \Gamma$. We now conclude that the image of the map (6.17) is the whole $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.

The fact that this map is a *-homomorphism also follows directly from the commutation relation proved above. $\square$

6.2 Representation theory

In this section we develop the representation theory of crossed products by Hecke pairs. We will introduce the notion of a *covariant pre-representation* and show that there is a bijective correspondence between covariant pre-representations and representations of the crossed product, in a similar fashion to the theory of crossed products by groups.

Recall from Proposition 4.2.16 that every nondegenerate *-representation $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ extends uniquely to a *-representation

$$\widetilde{\pi} : M_B(C_c(\mathcal{A}/\Gamma)) \to B(\mathscr{H}) \,.$$

We will use the notation $\widetilde{\pi}$ to denote this extension throughout this section, many times without any reference. Since $C_c(X^0/\Gamma)$ is a BG^*-algebra we naturally have $C_c(X^0/\Gamma) \subseteq M_B(C_c(\mathcal{A}/\Gamma))$.

Definition 6.2.1. Let π be a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$ on a Hilbert space $\mathscr{H}$ and $\widetilde{\pi}$ its unique extension to a *-representation of $M_B(C_c(\mathcal{A}/\Gamma))$. Let μ be a unital pre-*-representation of $\mathcal{H}(G,\Gamma)$ on the inner product space $\mathscr{W} := \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We say that (π,μ) is a *covariant pre-*-representation* if

$$\pi(a_{x\Gamma})\mu(\Gamma g\Gamma) \in B(\mathscr{W})\,, \tag{6.18}$$

for all $a \in \mathcal{A}_x$, $x \in X$ and $g \in G$, and the following equality

$$\mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\mu(\Gamma s\Gamma) \; = \tag{6.19}$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \; \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\,\widetilde{\pi}(a_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})\,,$$

holds on $L(\mathscr{W})$, for all $g, s \in G$ and $x \in X$.

Condition (6.18) is a rather technical requirement, but a necessary one, so that several expressions we shall use are well defined. Condition (6.19), on the other hand, is more natural and simply says that the pair (π,μ) must preserve the structure of products of the form $\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma$, when expressed in terms of the canonical spanning set of elements of the form $b_{y\Gamma} * \Gamma d\Gamma * 1_{\mathbf{s}(y)d\Gamma}$, as explicitly described in Corollary 6.1.19.

The reader should note the similarity between our definition of a covariant pre-*-representation and the *covariant pairs* of an Huef, Kaliszewski and Raeburn in [19, Definition 1.1]. Their notion of covariant pairs served as a motivation for us and is actually a particular case of our Definition 6.2.1, as we shall see in Chapter 10.

The striking feature that we actually have to consider pre-representations of $\mathcal{H}(G,\Gamma)$, and not just representations, was not present in the theory of crossed products by groups because a group algebra $\mathbb{C}(G)$ of a discrete group is always a BG^*-algebra and therefore all of its pre-representations come from true representations (see further Remark 6.2.5).

It will be useful to distinguish between covariant pre-*-representations and covariant *-representations, so we will treat them in separate definitions. As will be discussed below we will see covariant *-representations as a particular type of covariant pre-*-representations.

Definition 6.2.2. Let π be a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$ on a Hilbert space $\mathscr{H}$ and μ a unital *-representation of $\mathcal{H}(G,\Gamma)$ on $\mathscr{H}$. We say that (π,μ) is a *covariant *-representation* if equality (6.19) holds in $B(\mathscr{H})$ for all $g,s\in G$ and $x\in X$.

Lemma 6.2.3. *Let (π,μ) be a covariant *-representation on a Hilbert space $\mathscr{H}$. Then μ leaves the subspace $\mathscr{W} := \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ invariant.*

Proof: Consider elements of the form $\pi(a_{x\Gamma})\xi$, whose span gives $\mathscr{W}$. Using the fact that μ is unital and the covariance relation (6.19) we see that

$$\begin{aligned} & \mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\xi = \\ = \; & \mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\mu(\Gamma)\xi \\ = \; & \sum_{[u]\in\Gamma g^{-1}\Gamma/\Gamma}\ \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,e}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},e}}{L(u^{-1})}\,\widetilde{\pi}(a_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma\Gamma})\xi\,. \end{aligned}$$

Hence, $\mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\xi\in\mathscr{W}$, and consequently $\mu(\Gamma g\Gamma)$ leaves $\mathscr{W}$ invariant. This finishes the proof. $\square$

From a covariant *-representation (π,μ) one can obtain canonically a covariant pre-*-representation (π,μ), just by restricting μ to the dense subspace $\mathscr{W} := \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ (which is an invariant subspace by Lemma 6.2.3). So we can regard covariant *-representations as a special kind of covariant pre-*-representations: they are exactly those for which μ is normed. As we shall see later in Example 6.3.1, there are covariant pre-*-representations which are not covariant *-representations, thus in general the latter form a proper subclass of the former. We shall also see examples where they actually coincide.

Remark 6.2.4. Equivalently, one could define covariant (pre-)*-representation using any other of the equalities in Corollary 6.1.19 and substituting with the appropriate (pre-)*-representations. It is easy to see, using completely analogous arguments as in the proof of Corollary 6.1.19 or Theorem 6.1.17, that all three expressions yield the same result.

Remark 6.2.5. Even though it might not be entirely clear from the start, when Γ is a normal subgroup of G the definition of a covariant pre-representation is nothing but the usual definition of a covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$. We recall that a covariant representation of $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$ is a pair (π, U) consisting of a nondegenerate *-representation π of $C_c(\mathcal{A}/\Gamma)$ and a unitary representation U of G/Γ satisfying the relation

$$\pi(\alpha_{g\Gamma}(f)) = U_{g\Gamma}\pi(f)U_{g^{-1}\Gamma}\,,$$

for all $f \in C_c(\mathcal{A}/\Gamma)$ and $g\Gamma \in G/\Gamma$. Now, as it is well known, every unitary representation U of G/Γ is associated in a canonical way to a unital *-representation μ of the group algebra $\mathbb{C}(G/\Gamma)$, so that we can write the covariance condition as $\pi(\alpha_{g\Gamma}(f)) = \mu(g\Gamma)\pi(f)\mu(g^{-1}\Gamma)$. As a consequence we have that for any $g\Gamma, s\Gamma \in G/\Gamma$, $x \in X$ and $a \in \mathcal{A}_x$:

$$\mu(g\Gamma)\pi(a_{x\Gamma})\mu(s\Gamma) = \pi(a_{xg^{-1}\Gamma})\mu(g^{-1}s\Gamma)\,.$$

We want to check that covariant representations of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$ are the same as covariant pre-*-representations as in Definition 6.2.1.

Given a covariant pre-*-representation (π, μ) on some Hilbert space $\mathscr{H}$ in the sense of Definition 6.2.1, we have that μ is a pre-*-representation of $\mathbb{C}(G/\Gamma)$, which is normed since any group algebra of a discrete group is a BG^*-algebra, and thus we can see μ as a true *-representation on $\mathscr{H}$. We then have that

$$\begin{aligned}
& \mu(g\Gamma)\pi(a_{x\Gamma})\mu(g^{-1}\Gamma) \\
= \; & \mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\mu(\Gamma g^{-1}\Gamma) \\
= \; & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma g^{-1}\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\,\widetilde{\pi}(a_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma}) \\
= \; & \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}} N^{\mathbf{s}(x)\gamma}_{g,g^{-1}}\,\widetilde{\pi}(a_{xg^{-1}\Gamma})\,\mu(gg^{-1}\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)g^{-1}\Gamma}) \\
= \; & \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}} N^{\mathbf{s}(x)\gamma}_{g,g^{-1}}\,\widetilde{\pi}(a_{xg^{-1}\Gamma}\cdot 1_{\mathbf{s}(x)g^{-1}\Gamma}) \\
= \; & \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}} N^{\mathbf{s}(x)\gamma}_{g,g^{-1}}\,\pi(a_{xg^{-1}\Gamma})\,.
\end{aligned}$$

It is clear from the normality of Γ that $E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}$ consists only of the class $[e]$ and moreover $N^{\mathbf{s}(x)}_{g,g^{-1}} = 1$, so that

$$\mu(g\Gamma)\pi(a_{x\Gamma})\mu(g^{-1}\Gamma) \;=\; \pi(a_{xg^{-1}\Gamma})\,.$$

By linearity it follows that $\mu(g\Gamma)\pi(f)\mu(g^{-1}\Gamma) = \pi(\alpha_{g\Gamma}(f))$ for any $f \in C_c(\mathcal{A}/\Gamma)$. Thus, with U being the unitary representation of G/Γ associated to μ, we see that (π, U) is covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$.

For the other direction, let (π, U) be a covariant representation (π, U) of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$ and let μ be the *-representation associated to U, which we restrict to the inner product space $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We want to prove that (π, μ) is a covariant pre-*-representation in the sense of Definition 6.2.1. Condition (6.18) is automatically satisfied since μ is a *-representation. Let us now check condition (6.19). We have

$$\begin{aligned}
&\mu(g\Gamma)\pi(a_{x\Gamma})\mu(s\Gamma)\\
=\;&\mu(g\Gamma)\pi(a_{x\Gamma})\mu(g^{-1}\Gamma)\mu(gs\Gamma)\\
=\;&\pi(a_{xg^{-1}\Gamma})\mu(gs\Gamma)\\
=\;&\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}}\;\sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\,\widetilde{\pi}(a_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})\,,
\end{aligned}$$

where the last equality is obtained following analogous computations as those above. Thus, (π, μ) is a covariant pre-*-representation in the sense of Definition 6.2.1.

The following result makes it clear that some of the relations we have inside the crossed product (see Proposition 6.1.16) are preserved upon taking covariant pre-*-representations. This is expected since, as we stated before, we will prove that covariant pre-representations give rise to representations of the crossed product, and this result is the first step in that direction:

Proposition 6.2.6. *Let (π, μ) be a covariant pre-*-representation. The following two equalities hold:*

$$\widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{xg\Gamma}) \;=\; \widetilde{\pi}(a_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,. \tag{6.20}$$

$$\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma}) = \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma})\,. \tag{6.21}$$

Proof: Since (π, μ) is a covariant pre-*-representation we have

$$\begin{aligned}
\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma}) &= \mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma})\mu(\Gamma) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} N^{\mathbf{s}(x)\gamma}_{g,e}\, \widetilde{\pi}(a_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\gamma\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(a_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\,,
\end{aligned}$$

where the last equality comes from the fact that $n^{\mathbf{s}(x)\gamma}_{g,e} = 1 = d^{\mathbf{s}(x)\gamma}_{g,e}$, and thus $N^{\mathbf{s}(x)\gamma}_{g,e} = 1$. From this it follows that

$$\begin{aligned}
\widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma}) &= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma})\widetilde{\pi}(a_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma}\cdot a_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\,.
\end{aligned}$$

Now the product $1_{\mathbf{r}(x)g^{-1}\Gamma}\cdot a_{x\gamma g^{-1}\Gamma}$ is nonzero only when $\mathbf{r}(x)g^{-1}\Gamma = \mathbf{r}(x)\gamma g^{-1}\Gamma$, from which one readily concludes that $\mathbf{r}(x)\gamma \in \mathbf{r}(x)g^{-1}\Gamma g$. Since one trivially has $\mathbf{r}(x)\gamma \in \mathbf{r}(x)\Gamma$ we conclude that

$$\mathbf{r}(x)\gamma \in \mathbf{r}(x)\Gamma \cap \mathbf{r}(x)g^{-1}\Gamma g\,,$$

and by the Γ-intersection property we have $\mathbf{r}(x)\gamma \in \mathbf{r}(x)\Gamma^{g^{-1}}$. From Proposition 1.2.2 this means that $[\gamma] = [e]$ in $E^{\mathbf{r}(x)}_{g^{-1},e}$. We recall that $E^{\mathbf{r}(x)}_{g^{-1},e} = \mathcal{S}_{\mathbf{r}(x)}\backslash\Gamma/\Gamma^{g^{-1}}$, and since $\Gamma^{g^{-1}} \subseteq \Gamma$ we have by Proposition 1.2.1 that $[\gamma] \to [\gamma]$ defines a canonical bijection between $E^{\mathbf{r}(x)}_{g^{-1},e}$ and $(\mathcal{S}_{\mathbf{r}(x)}\cap\Gamma)\backslash\Gamma/\Gamma^{g^{-1}}$. Since the G-action is Γ-good we necessarily have $\mathcal{S}_{\mathbf{s}(x)}\cap\Gamma = \mathcal{S}_x\cap\Gamma = \mathcal{S}_{\mathbf{r}(x)}\cap\Gamma$, and therefore using Proposition 1.2.1 one more time we can say that $E^{\mathbf{r}(x)}_{g^{-1},e} = E^{\mathbf{s}(x)}_{g^{-1},e}$. Hence, we can say that $[\gamma] = [e]$ in $E^{\mathbf{s}(x)}_{g^{-1},e}$. We conclude that

$$\begin{aligned}
\widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma}) &= \widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma}\cdot a_{xg^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma}) \\
&= \widetilde{\pi}(a_{xg^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\,.
\end{aligned}$$

Since the last expression is valid for any $x \in X$, if we take x to be xg we obtain the desired equality (6.20):

$$\widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{xg\Gamma}) = \widetilde{\pi}(a_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,.$$

Let us now prove equality (6.21). Using the equality in beginning of this proof and equality (6.20) which we have just proven, we get precisely

$$\begin{aligned}
\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma}) &= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(a_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(a_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x\gamma g^{-1})g\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\gamma g^{-1}g\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma})\,.
\end{aligned}$$

This finishes the proof. □

The passage from a covariant pre-representation (π,μ) to a representation of $C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma$ is done via the so-called *integrated form* $\pi\times\mu$, which we now describe:

Definition 6.2.7. Let (π,μ) be a covariant pre-*-representation on a Hilbert space $\mathscr{H}$. We define the *integrated form* of (π,μ) as the function $\pi\times\mu$: $C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma\to B(\mathscr{H})$ defined by

$$[\pi\times\mu](f) := \sum_{[g]\in\Gamma\backslash G/\Gamma}\ \sum_{x\Gamma^g\in X/\Gamma^g} \widetilde{\pi}\Big(\big(f(g\Gamma)(x\Gamma^g)\big)_{x\Gamma}\Big)\,\mu(\Gamma g\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,.$$

Remark 6.2.8. For f of the form $f=a_{x\Gamma}*\Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ we have

$$[\pi\times\mu](f)=\widetilde{\pi}(a_{x\Gamma})\,\mu(\Gamma g\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,.$$

Moreover, from equality (6.20), for f' of the form $f'=1_{\mathbf{r}(x)\Gamma}*\Gamma g\Gamma * a_{xg\Gamma}$ we have

$$[\pi\times\mu](f')=\widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\,\mu(\Gamma g\Gamma)\,\widetilde{\pi}(a_{xg\Gamma})\,.$$

Proposition 6.2.9. *The integrated form $\pi\times\mu$ of a covariant pre-*-representation (π,μ) is a well-defined nondegenerate *-representation.*

Proof: First we need to check that the expression that defines $[\pi \times \mu](f)$ for a given $f \in C_c(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is well-defined. This is proven in an entirely analogous way as in the proof that the expression (6.5) in Proposition 6.1.14 is well-defined. Secondly, we need to show that $[\pi \times \mu](f)$ makes sense as an element of $B(\mathscr{H})$. Surely, $[\pi \times \mu](f) \in L(\mathscr{W})$, but by definition of a covariant pre-*-representation we have

$$\widetilde{\pi}\Big(f(g\Gamma)(x\Gamma^g)\big)_{x\Gamma}\Big)\mu(\Gamma g\Gamma) \in B(\mathscr{W}) ,$$

for all $g\Gamma \in G/\Gamma$. Thus, it follows that $[\pi \times \mu](f) \in B(\mathscr{W})$, and therefore $[\pi \times \mu](f)$ admits a unique extension to $B(\mathscr{H})$.

Now, it is obvious that $\pi \times \mu$ is a linear transformation. Let us check that it preserves the involution. It is then enough to check it for elements of the form $f = a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$. Since (π, μ) is a covariant pre-*-representation we have, by Propositions 6.2.6 and 6.1.16,

$$\begin{aligned}
\big([\pi \times \mu](f)\big)^* &= \Delta(g)\, \widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}(a^*_{x^{-1}\Gamma}) \\
&= \Delta(g)\, \widetilde{\pi}(1_{\mathbf{r}(x^{-1})g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}(a^*_{x^{-1}gg^{-1}\Gamma}) \\
&= \Delta(g)\, \widetilde{\pi}(a^*_{x^{-1}g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x^{-1})gg^{-1}\Gamma}) \\
&= \Delta(g)\, \widetilde{\pi}(a^*_{x^{-1}g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x^{-1})\Gamma}) \\
&= [\pi \times \mu]\,(\Delta(g)\; a^*_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1})\Gamma}) \\
&= [\pi \times \mu]\,(f^*)\,.
\end{aligned}$$

Let us now prove that $\pi \times \mu$ preserves products. We will start by proving that

$$[\pi \times \mu](f_1 * f_2) = [\pi \times \mu](f_1)\, [\pi \times \mu](f_2)\,, \tag{6.22}$$

for $f_1 := a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ and $f_2 := b_{y\Gamma} * \Gamma s\Gamma * 1_{\mathbf{s}(y)s\Gamma}$. Let us compute the expression on the left side of (6.22). First, we notice that for the product $f_1 * f_2$ to be non-zero one must have $\mathbf{r}(y) \in \mathbf{s}(x)g\Gamma$, and in this case we obtain

$$f_1 * f_2 = a_{x\Gamma} * \Gamma g\Gamma * b_{y\Gamma} * \Gamma s\Gamma * 1_{\mathbf{s}(y)s\Gamma}$$

which by Corollary 6.1.19 gives

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E_{u,v}^{\mathbf{s}(y)}} \frac{\Delta(g) N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)} \big(a_{x\Gamma} * b_{y\gamma u\Gamma} * \Gamma u^{-1} v\Gamma * 1_{\mathbf{s}(y)\gamma v\Gamma} * 1_{\mathbf{s}(y)s\Gamma}\big)$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{\substack{[\gamma]\in E_{u,v}^{\mathbf{s}(y)} \\ \mathbf{s}(y)s\Gamma=\mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)} \big(a_{x\Gamma} * b_{y\gamma u\Gamma} * \Gamma u^{-1} v\Gamma * 1_{\mathbf{s}(y)\gamma v\Gamma}\big)$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{\substack{[\gamma]\in E_{u,v}^{\mathbf{s}(y)} \\ \mathbf{s}(y)s\Gamma=\mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N_{u^{-1},v}^{s(x)\gamma}}{L(u^{-1}v)} \big(a_{x\Gamma} * b_{y\gamma u\Gamma} * \Gamma u^{-1} v\Gamma * 1_{\mathbf{s}(y\gamma u)u^{-1}v\Gamma}\big)$$

The product $a_{x\Gamma} * b_{y\gamma u\Gamma}$ is always either zero or of the form $c_{(x\theta)(y\gamma u)\Gamma}$, for some $\theta \in \Gamma$ and $c \in \mathcal{A}_{(x\theta)(y\gamma u)}$. The point is that $\mathbf{s}\big((x\theta)(y\gamma u)\big) = \mathbf{s}(y\gamma u)$, so that each non-zero summand in the last sum above is actually of the form

$$c_{z\Gamma} * \Gamma d\Gamma * 1_{\mathbf{s}(z)d\Gamma}\,,$$

for appropriate $c \in \mathcal{A}_z$, $z \in X$ and $d \in G$. Thus, by linearity of $\pi \times \mu$ and Remark 6.2.8 we obtain

$$[\pi \times \mu](f_1 * f_2) =$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{\substack{[\gamma]\in E_{u,v}^{\mathbf{s}(y)} \\ \mathbf{s}(y)s\Gamma=\mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)}\, \widetilde{\pi}(a_{x\Gamma} \cdot b_{y\gamma u\Gamma})\, \mu(\Gamma u^{-1} v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)\gamma v\Gamma})\,.$$

Let us now compute the expression on the right side of (6.22). We have

$$[\pi \times \mu](f_1)\, [\pi \times \mu](f_2) = \ \widetilde{\pi}(a_{x\Gamma})\, \mu(\Gamma g\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\, \widetilde{\pi}(b_{y\Gamma})\, \mu(\Gamma s\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)s\Gamma})\,.$$

For $1_{\mathbf{s}(x)g\Gamma} \cdot b_{y\Gamma}$ to be non-zero we must have $\mathbf{r}(y) \in \mathbf{s}(x)g\Gamma$, and in this case we obtain, using the definition of a covariant pre-*-representation,

$$[\pi \times \mu](f_1)\, [\pi \times \mu](f_2) =$$

$$= \ \widetilde{\pi}(a_{x\Gamma})\, \mu(\Gamma g\Gamma)\, \widetilde{\pi}(b_{y\Gamma})\, \mu(\Gamma s\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)s\Gamma})$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E_{u,v}^{\mathbf{s}(y)}}} \frac{\Delta(g) N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)} \widetilde{\pi}(a_{x\Gamma})\, \widetilde{\pi}(b_{y\gamma u\Gamma})\, \mu(\Gamma u^{-1} v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)\gamma v\Gamma})\, \widetilde{\pi}(1_{\mathbf{s}(y)s\Gamma})$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{\substack{[\gamma]\in E_{u,v}^{\mathbf{s}(y)} \\ \mathbf{s}(y)s\Gamma=\mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)} \widetilde{\pi}(a_{x\Gamma} \cdot b_{y\gamma u\Gamma})\, \mu(\Gamma u^{-1} v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)\gamma v\Gamma})\,.$$

Hence, we have proven equality (6.22) for the special case of f_1 and f_2 being $f_1 := a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ and $f_2 := b_{y\Gamma} * \Gamma s\Gamma * 1_{\mathbf{s}(y)s\Gamma}$. Using this we will now show that equality (6.22) holds for any $f_1, f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. In fact, by Proposition 6.1.14, f_1 and f_2 can be written as sums

$$f_1 = \sum_i v_i \,, \qquad\qquad f_1 = \sum_j w_j \,,$$

where each v_i and w_j is of the form $a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$, for some $g\Gamma \in G/\Gamma$, $x \in X$ and $a \in \mathcal{A}_x$. Since $\pi \times \mu$ is a linear mapping we have

$$\begin{aligned}
[\pi \times \mu](f_1 * f_2) &= [\pi \times \mu]\Big(\big(\sum_i v_i\big) * \big(\sum_j w_j\big)\Big) \\
&= [\pi \times \mu]\Big(\sum_{i,j} v_i * w_j\Big) \\
&= \sum_{i,j} [\pi \times \mu](v_i * w_j) \,,
\end{aligned}$$

and by the special case of equality (6.22) we have just proven we get

$$\begin{aligned}
[\pi \times \mu](f_1 * f_2) &= \sum_{i,j} [\pi \times \mu](v_i)[\pi \times \mu](w_j) \\
&= \Big(\sum_i [\pi \times \mu](v_i)\Big)\Big(\sum_j [\pi \times \mu](w_j)\Big) \\
&= [\pi \times \mu]\big(\sum_i v_i\big)[\pi \times \mu]\big(\sum_j w_j\big) \\
&= [\pi \times \mu](f_1)[\pi \times \mu](f_2) \,.
\end{aligned}$$

Hence, $\pi \times \mu$ is a *-representation. To finish the proof we now only need to show that $\pi \times \mu$ is nondegenerate. The restriction of $\pi \times \mu$ to the *-subalgebra $C_c(\mathcal{A}/\Gamma)$ is precisely the representation π. Since π is assumed to be nondegenerate it follows that $\pi \times \mu$ must be nondegenerate as well. $\square$

The next result shows how from a representation of the crossed product one can naturally form a covariant pre-representation.

Proposition 6.2.10. *Let $\Phi : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ be a nondegenerate *-representation. Consider the pair $(\Phi|, \mu_\Phi)$ defined by*

- *$\Phi|$ is the restriction of Φ to $C_c(\mathcal{A}/\Gamma)$.*

- *Let $\widetilde{\Phi}$ be the extension of Φ to a pre-*-representation (via Proposition 4.2.13) of $M(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)$ on the inner product space $\Phi(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)\mathscr{H}$. We define μ_Φ to be the restriction of $\widetilde{\Phi}$ to $\mathcal{H}(G,\Gamma)$.*

The pair $(\Phi|, \mu_\Phi)$ is a covariant pre-*-representation.

We will need some preliminary facts and lemmas in order to prove Proposition 6.2.10.

Let $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\pi}$ its extension to $M_B(C_c(\mathcal{A}/\Gamma))$. For any unit $u \in X^0$ the operator $\widetilde{\pi}(1_{u\Gamma}) \in B(\mathscr{H})$ is a projection, and therefore $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ is a Hilbert subspace. The fiber $\mathcal{A}_{u\Gamma}$ is a C^*-algebra which we can naturally identify with the *-subalgebra

$$\{a_{u\Gamma} \in C_c(\mathcal{A}/\Gamma) : a \in \mathcal{A}_{u\Gamma}\} \subseteq C_c(\mathcal{A}/\Gamma)\,,$$

under the identification given by

$$\mathcal{A}_{u\Gamma} \ni a \;\longleftrightarrow\; a_{u\Gamma} \in C_c(\mathcal{A}/\Gamma)\,.$$

The *-representation $\widetilde{\pi}$ when restricted to $\mathcal{A}_{u\Gamma}$, under the above identification, leaves the subspace $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ invariant, because

$$\widetilde{\pi}(a_{u\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = \widetilde{\pi}(a_{u\Gamma})\xi = \widetilde{\pi}(1_{u\Gamma})\widetilde{\pi}(a_{u\Gamma})\xi\,.$$

The following lemma assures that this restriction is nondegenerate.

Lemma 6.2.11. *Let $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\pi}$ its unique extension to $M_B(C_c(\mathcal{A}/\Gamma))$. The *-representation of $\mathcal{A}_{u\Gamma}$ on the Hilbert space $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$, as above, is nondegenerate.*

Proof: Let $\widetilde{\pi}(1_{u\Gamma})\xi$ be an element of $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ such that

$$\widetilde{\pi}(a_{u\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = 0\,,$$

for all $a \in \mathcal{A}_{u\Gamma}$. We want to prove that $\widetilde{\pi}(1_{u\Gamma})\xi = 0$. To see this, let $x \in X$ and $b \in \mathcal{A}_{x\Gamma}$. We have two alternatives: either $\mathbf{s}(x)\Gamma \neq u\Gamma$ or $\mathbf{s}(x)\Gamma = u\Gamma$. In the first case we see that

$$\widetilde{\pi}(b_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = \widetilde{\pi}(b_{x\Gamma}\cdot 1_{u\Gamma})\xi = 0\,,$$

whereas for the second we see that

$$\begin{aligned}\|\widetilde{\pi}(b_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\|^2 &= \langle\widetilde{\pi}(b_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\,,\,\widetilde{\pi}(b_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\rangle\\ &= \langle\widetilde{\pi}((b^*b)_{\mathbf{s}(x)\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\,,\,\widetilde{\pi}(1_{u\Gamma})\xi\rangle\\ &= \langle\widetilde{\pi}((b^*b)_{u\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\,,\,\widetilde{\pi}(1_{u\Gamma})\xi\rangle\\ &= 0\,,\end{aligned}$$

by assumption. Thus, in any case we have $\widetilde{\pi}(b_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = 0$ for all $x \in X$ and $b \in \mathcal{A}_{x\Gamma}$. By nondegeneracy of π, this implies that $\widetilde{\pi}(1_{u\Gamma})\xi = 0$, as we wanted to prove. □

Lemma 6.2.12. *If $\Phi : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ is a nondegenerate *-representation, then its restriction to $C_c(\mathcal{A}/\Gamma)$ is also nondegenerate.*

Proof: Let $\xi \in \mathscr{H}$ be such that $\Phi(C_c(\mathcal{A}/\Gamma))\,\xi = \{0\}$. We want to show that $\xi = 0$. Since Φ is nondegenerate, it is then enough to prove that $\Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\,\xi = \{0\}$. Thus, by virtue of Proposition 6.1.16, it suffices to prove that $\Phi(1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{xg\Gamma})\xi = 0$ for all $g \in G$, $x \in X$, $a \in \mathcal{A}_x$. We have

$$\begin{aligned}&\|\Phi(1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{xg\Gamma})\xi\|^2 = \\ =\ &\Delta(g)\,\langle\Phi(a^*_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma} * 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{xg\Gamma})\xi\,,\,\xi\rangle\\ =\ &\Delta(g)\,\langle\Phi(a^*_{x^{-1}g\Gamma})\Phi(\Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{xg\Gamma})\xi\,,\,\xi\rangle\\ =\ &\Delta(g)\,\langle\Phi(\Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * a_{xg\Gamma})\xi\,,\,\Phi(a_{xg\Gamma})\xi\rangle\\ =\ &0\,.\end{aligned}$$

Hence $\xi = 0$ and therefore Φ restricted to $C_c(\mathcal{A}/\Gamma)$ is nondegenerate. □

Lemma 6.2.13. *Let $\Phi : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\Phi}$ its unique extension to $M_B(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$ (via Proposition 4.2.16). Let $\Phi|$ be the restriction of Φ to $C_c(\mathcal{A}/\Gamma)$ and $\widetilde{\Phi|}$ its unique extension to $M_B(C_c(\mathcal{A}/\Gamma))$. We have that*

$$\widetilde{\Phi}(f) = \widetilde{\Phi|}(f)\,,$$

for all $f \in C_c(X^0/\Gamma)$. In other words, the two *-representations $\widetilde{\Phi}$ and $\widetilde{\Phi|}$ are the same in $C_c(X^0/\Gamma)$.

Proof: By Lemma 6.2.12 the subspace $\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ is dense in $\mathscr{H}$, so that it is enough to check that $\widetilde{\Phi}(f)\Phi(f_2)\xi = \widetilde{\Phi|}(f)\Phi(f_2)\xi$, for all $f_2 \in C_c(\mathcal{A}/\Gamma)$ and $\xi \in \mathscr{H}$. By definition of the extension $\widetilde{\Phi}$ (see Proposition 4.2.16) we have

$$\widetilde{\Phi}(f)\Phi(f_2)\xi \;=\; \Phi(f * f_2)\,,$$

where $f * f_2$ is the product of f and f_2, which lies inside $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. Since both f and f_2 are elements of $B(\mathcal{A}, G, \Gamma)$ we see the product $f * f_2$ as taking place in $B(\mathcal{A}, G, \Gamma)$. By definition of the embeddings of $C_c(X^0/\Gamma)$ and $C_c(\mathcal{A}/\Gamma)$ in $B(\mathcal{A}, G, \Gamma)$ we have that $f * f_2$ is nothing but the element $f \cdot f_2$, where the product is just the product of f and f_2 inside $M(C_c(\mathcal{A}))$. As we observed in Section 5.3, this product is exactly same as the product of f and f_2 in $M(C_c(\mathcal{A}/\Gamma))$. Thus, the following computation makes sense:

$$\begin{aligned}\widetilde{\Phi}(f)\Phi(f_2)\xi &= \Phi(f * f_2)\xi \;=\; \Phi(f \cdot f_2)\xi \\ &= \Phi|(f \cdot f_2)\xi \;=\; \widetilde{\Phi|}(f)\Phi|(f_2)\xi\,.\end{aligned}$$

This finishes the proof. □

Lemma 6.2.14. *Let $\Phi : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ be a nondegenerate *-representation. We have that*

$$\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \;=\; \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H}\,.$$

Proof: The inclusion $\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \subseteq \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H}$ is obvious. To check the converse inclusion it is enough to prove that

$$\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi \in \Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}\,,$$

for all $x \in X$, $a \in \mathcal{A}_x$, $g \in G$ and $\xi \in \mathscr{H}$. Let $\widetilde{\Phi} : M_B(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) \to B(\mathscr{H})$ be the unique extension of Φ to a *-representation of $M_B(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$, as in Proposition 4.2.16. We then get

$$\begin{aligned}\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi &= \Phi(1_{\mathbf{r}(x)\Gamma} * a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi \\ &= \widetilde{\Phi}(1_{\mathbf{r}(x)\Gamma})\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi\,,\end{aligned}$$

i.e. $\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi \in \widetilde{\Phi}(1_{\mathbf{r}(x)\Gamma})\mathscr{H}$.

Let us denote the restriction of Φ to $C_c(\mathcal{A}/\Gamma)$ by $\Phi|$, which is a nondegenerate *-representation by Lemma 6.2.12. By Lemma 6.2.13 we can also write

$$\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi \in \widetilde{\Phi|}(1_{\mathbf{r}(x)\Gamma})\mathscr{H} ,$$

where $\widetilde{\Phi|}$ is the unique extension of $\Phi|$ to $M_B(C_c(\mathcal{A}/\Gamma))$.

By Lemma 6.2.11, we know that $\Phi|$ yields a nondegenerate *-representation of $\mathcal{A}_{\mathbf{r}(x)\Gamma}$ on $\widetilde{\Phi|}(1_{\mathbf{r}(x)\Gamma})\mathscr{H}$. Since the fiber $\mathcal{A}_{\mathbf{r}(x)\Gamma}$ is a C^*-algebra we have, by the general version of Cohen's factorization theorem ([34, Theorem 5.2.2]), that there exists $c \in \mathcal{A}_{\mathbf{r}(x)\Gamma}$ and $\eta \in \widetilde{\Phi}(1_{\mathbf{r}(x)\Gamma})\mathscr{H}$ such that

$$\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi = \Phi|(c_{\mathbf{r}(x)\Gamma})\eta = \Phi(c_{\mathbf{r}(x)\Gamma})\eta \,.$$

Hence we conclude that $\Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H} \subseteq \Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. $\square$

Proof of Proposition 6.2.10: First of all, by Lemma 6.2.12, $\Phi|$ is indeed a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$. Secondly, from Lemma 6.2.14, we have

$$\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \;=\; \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H} \,.$$

Thus, μ_Φ is a pre-*-representation of $\mathcal{H}(G,\Gamma)$ on $\mathscr{W} := \Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We now only need to check covariance. First, since

$$\begin{aligned} \Phi|(a_{x\Gamma})\mu_\Phi(\Gamma g\Gamma)\,\omega &= \Phi(a_{x\Gamma})\widetilde{\Phi}(\Gamma g\Gamma)\,\omega \\ &= \Phi(a_{x\Gamma} * \Gamma g\Gamma)\,\omega \,, \end{aligned}$$

with $\omega \in \mathscr{W}$, we conclude that $\Phi|(a_{x\Gamma})\mu_\Phi(\Gamma g\Gamma) \in B(\mathscr{W})$, for all $x \in X$, $a \in \mathcal{A}_x$ and $g \in G$. Moreover we have

$$\begin{aligned} &\mu_\Phi(\Gamma g\Gamma)\Phi|(a_{x\Gamma})\mu_\Phi(\Gamma s\Gamma)\,\omega \;= \\ =\;& \widetilde{\Phi}(\Gamma g\Gamma)\widetilde{\Phi}(a_{x\Gamma})\widetilde{\Phi}(\Gamma s\Gamma)\,\omega \\ =\;& \widetilde{\Phi}(\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma)\,\omega \\ =\;& \widetilde{\Phi}\Big(\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E_{u,v}^{\mathbf{s}(x)}} \frac{\Delta(g)N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)}\, a_{x\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\Big)\,\omega \\ =\;& \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E_{u,v}^{\mathbf{s}(x)}} \frac{\Delta(g)N_{u^{-1},v}^{\mathbf{s}(x)\gamma}}{L(u^{-1}v)}\, \widetilde{\Phi}(a_{x\gamma u\Gamma})\widetilde{\Phi}(\Gamma u^{-1}v\Gamma)\widetilde{\Phi}(1_{\mathbf{s}(x)\gamma v\Gamma})\,\omega \,. \end{aligned}$$

Denoting by $\widetilde{\Phi|}$ the unique extension of $\Phi|$ to $M_B(C_c(\mathcal{A}/\Gamma))$ we have, by Lemma 6.2.13, that

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g) N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \widetilde{\Phi|}(a_{x\gamma u\Gamma})\mu_\Phi(\Gamma u^{-1}v\Gamma)\widetilde{\Phi|}(1_{\mathbf{s}(x)\gamma v\Gamma})\,\omega\,,$$

for all $\omega \in \mathscr{W}$. This finishes the proof. $\square$

Theorem 6.2.15. *There is a bijective correspondence between nondegenerate *-representations of $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ and covariant pre-*-representations. This bijection is given by $(\pi, \mu) \longmapsto \pi \times \mu$, with inverse given by $\Phi \longmapsto (\Phi|, \mu_\Phi)$.*

Proof: We have to prove that the composition of these maps, in both orders, is the identity.

Let (π, μ) be a covariant pre-*-representation and $\pi \times \mu$ its integrated form. We want to show that

$$\big((\pi\times\mu)|,\, \mu_{\pi\times\mu}\big) = (\pi,\mu)\,.$$

By definition of the integrated form we readily have $(\pi \times \mu)| = \pi$. This also implies, via Lemma 6.2.12, that the inner product spaces on which μ and $\mu_{\pi\times\mu}$ are defined are actually the same. Thus, it remains to be checked that $\mu_{\pi\times\mu} = \mu$. Let $\pi(a_{x\Gamma})\xi$ be one of the generators of $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We have

$$\begin{aligned} & \mu_{\pi\times\mu}(\Gamma g\Gamma)\,\pi(a_{x\Gamma})\xi \;= \\ = \;& \widetilde{[\pi\times\mu]}(\Gamma g\Gamma)\,\pi(a_{x\Gamma})\xi \\ = \;& [\pi\times\mu](\Gamma g\Gamma * a_{x\Gamma})\xi\,, \end{aligned}$$

and using Proposition 6.1.16, Remark 6.2.8 and Proposition 6.2.6 we obtain

$$\begin{aligned} = \;& [\pi\times\mu]\Big(\sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} 1_{\mathbf{r}(x)\gamma g\Gamma} * \Gamma g\Gamma * a_{x\Gamma}\Big)\xi \\ = \;& \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(a_{x\Gamma})\xi \\ = \;& \mu(\Gamma g\Gamma)\,\pi(a_{x\Gamma})\xi \end{aligned}$$

Hence, we conclude that $\mu_{\pi\times\mu} = \mu$.

Now let Φ be a *-representation of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and $(\Phi|, \mu_\Phi)$ its corresponding covariant pre-*-representation. We want to prove that

$$\Phi| \times \mu_\Phi = \Phi\,.$$

Let $1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * a_{xg\Gamma}$ be one of the spanning elements of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and $\xi \in \mathscr{H}$. We have

$$[\Phi| \times \mu_\Phi]\,(1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * a_{xg\Gamma})\,\xi \;=\; \widetilde{\Phi|}(1_{\mathbf{r}(x)\Gamma})\mu_\Phi(\Gamma g \Gamma)\widetilde{\Phi|}(a_{xg\Gamma})\,\xi\,,$$

which by Lemma 6.2.13 gives that

$$\begin{aligned} &= \widetilde{\Phi}(1_{\mathbf{r}(x)\Gamma})\widetilde{\Phi}(\Gamma g \Gamma)\widetilde{\Phi}(a_{xg\Gamma})\,\xi \\ &= \Phi(1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * a_{xg\Gamma})\,\xi\,. \end{aligned}$$

Thus, $\Phi| \times \mu_\Phi = \Phi$. $\square$

We will now show, in Proposition 6.2.17 below, that the bijective correspondence between covariant pre-*-representations and nondegenerate *-representations of the crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ behaves as expected regarding unitary equivalence. First however we make the following remark/definition:

Remark 6.2.16. Let (π, μ) be a covariant pre-*-representation on a Hilbert space $\mathscr{H}$. If $\mathscr{H}_0$ is another Hilbert space and $U : \mathscr{H} \to \mathscr{H}_0$ is a unitary, then it is easily seen that $(U\pi U^*, U\mu U^*)$ is also a covariant pre-*-representation. We will henceforward say that two covariant pre-*-representations (π_1, μ_1) and (π_2, μ_2) are *unitarily equivalent* if there exists a unitary U between the underlying Hilbert spaces such that $(\pi_1, \mu_1) = (U\pi_2 U^*, U\mu_2 U^*)$.

Proposition 6.2.17. *Suppose that (π_1, μ_1) and (π_2, μ_2) are two covariant pre-*-representations. Then (π_1, μ_1) is unitarily equivalent to (π_2, μ_2) if and only if $\pi_1 \times \mu_1$ is unitarily equivalent to $\pi_2 \times \mu_2$.*

Proof: $(\Longrightarrow)$ This direction is straightforward from the definition of the integrated form and from the following computation, where U is a unitary which establishes an equivalence between (π_1, μ_1) and (π_2, μ_2) :

$$\begin{aligned} [U(\pi_1 \times \mu_1)U^*](a_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma}) &= U\pi_1(a_{x\Gamma})\mu_1(\Gamma g \Gamma)\widetilde{\pi_1}(1_{xg\Gamma})U^* \\ &= U\pi_1(a_{x\Gamma})U^*U\mu_1(\Gamma g \Gamma)U^*U\widetilde{\pi_1}(1_{xg\Gamma})U^* \\ &= [U\pi_1 U^* \times U\mu_1 U^*](a_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma}) \\ &= [\pi_2 \times \mu_2](a_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma})\,, \end{aligned}$$

.

($\Longleftarrow$) Suppose that $\pi_1 \times \mu_1$ and $\pi_2 \times \mu_2$ are unitarily equivalent and let U be a unitary which establishes this equivalence. Then, since π_1 and π_2 are just the restrictions of, respectively, $\pi_1 \times \mu_1$ and $\pi_2 \times \mu_2$ we automatically have that $U\pi_1 U^* = \pi_2$. To see that $U\mu_1 U^* = \mu_2$ we just note that U canonically establishes a unitary equivalence between the pre-*-representations $\widetilde{\pi_1 \times \mu_1}$ and $\widetilde{\pi_2 \times \mu_2}$. $\square$

6.3 More on covariant pre-*-representations

In the previous section we introduced the notion of covariant pre-*-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ (Definition 6.2.1) and a particular instance of these which we called covariant *-representations (Definition 6.2.2).

In this section we will see that the class of covariant pre-*-representations is in general strictly larger than the class of covariant *-representations. It is thus unavoidable, in general, to consider pre-representations of the Hecke algebra in the representation theory of crossed products by Hecke pairs. We shall also see, nevertheless, that in many interesting situations every covariant pre-*-representation is actually a covariant *-representation.

Example 6.3.1. Let (G, Γ) be a Hecke pair such that its corresponding Hecke algebra $\mathcal{H}(G, \Gamma)$ does not have an enveloping C^*-algebra (for example $(G, \Gamma) = (\mathbb{Z} \rtimes \mathbb{Z}_2, \mathbb{Z}_2)$). The fact that the Hecke algebra does not have an enveloping C^*-algebra implies that there is a sequence of *-representations $\{\mu_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}(G, \Gamma)$ on Hilbert spaces $\{\mathscr{H}_n\}_{n\in\mathbb{N}}$ and an element $f \in \mathcal{H}(G, \Gamma)$ such that $\|\mu_n(f)\| \to \infty$. Let $\mathscr{V}$ be the inner product space $\mathscr{V} := \bigoplus_{n\in\mathbb{N}} \mathscr{H}_n$ and $\mu : \mathcal{H}(G, \Gamma) \to L(\mathscr{V})$ the diagonal pre-*-representation

$$\mu := \bigoplus_{n\in\mathbb{N}} \mu_n\,,$$

which of course is not normed. Let $X = \{x_1, x_2, \dots\}$ be an infinite countable set, with the trivial groupoid structure, i.e. X is just a set. We consider the trivial action of G on X, i.e. all points are fixed by the action. Thus, the action is Γ-good and has the Γ-intersection property. Let $\mathcal{A}$ be the Fell bundle over X such that $\mathcal{A}_{x_n} = \mathbb{C}$ for all $x_n \in X$. It is clear that $\mathcal{A}$ has G-invariant fibers and that $\mathcal{A}/\Gamma = \mathcal{A}$. Moreover we have

$$C_c(\mathcal{A}/\Gamma) = C_c(X) = C_c(X^0/\Gamma)\,.$$

Let $\pi : C_c(X) \to B(\overline{\mathscr{V}})$ be the *-representation on the Hilbert space completion $\overline{\mathscr{V}}$ of $\mathscr{V}$ such that $\pi(1_{x_n})$ is the projection onto the subspace $\mathscr{H}_n$.

We claim that (π, μ) is a covariant pre-*-representation of $C_c(X) \times_\alpha^{alg} G/\Gamma$. To see this, first we notice that π is obviously nondegenerate and moreover $\pi(C_c(X))\overline{\mathscr{V}} = \mathscr{V}$, which is the inner product space where μ is defined. Next we notice that for every $x_n \in X$ and $g \in G$, the operators $\pi(1_{x_n})$ and $\mu(\Gamma g \Gamma)$ commute. Moreover, we have

$$\pi(1_{x_n})\mu(\Gamma g\Gamma)\pi(1_{x_n}) = \mu_n(\Gamma g\Gamma)\,,$$

on the subspace $\mathscr{H}_n$. Hence, the operator $\pi(1_{x_n})\mu(\Gamma g\Gamma)$ must be bounded on $\mathscr{V}$, and thus condition (6.18) is satisfied. Also we have

$$\begin{aligned}
\mu(\Gamma g\Gamma)\pi(1_{x_n})\mu(\Gamma s\Gamma) &= \mu(\Gamma g\Gamma)\mu(\Gamma s\Gamma)\pi(1_{x_n}) \\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)}\; \mu(\Gamma u^{-1}v\Gamma)\,\pi(1_{x_n}) \\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)}\; \pi(1_{x_n})\mu(\Gamma u^{-1}v\Gamma)\,\pi(1_{x_n}) \\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{x_n}_{u,v}} \frac{\Delta(g)N^{x_n\gamma}_{u^{-1},v}}{L(u^{-1}v)}\; \pi(1_{x_n\gamma u})\mu(\Gamma u^{-1}v\Gamma)\,\pi(1_{x_n\gamma v})\,,
\end{aligned}$$

where the last equality comes from the fact that since $\mathcal{S}_{x_n} = G$ we must have that $E^{x_n}_{u,v}$ consists only of the class $[e]$, $N^{x_n}_{u^{-1},v} = 1$ and also that $1_{x_n\gamma u} = 1_{x_n} = 1_{x_n\gamma v}$.

So we have established that (π, μ) is indeed a covariant pre-*-representation. Nevertheless, μ is not normed, so that (π, μ) is not a covariant *-representation.

It is worth noting that here we are in the conditions of Example 6.1.21, so that $C_c(X) \times_\alpha^{alg} G/\Gamma \;\cong\; C_c(X) \odot \mathcal{H}(G, \Gamma)$.

Example 6.3.1 shows that there can be more covariant pre-*-representations than covariant *-representations. Nevertheless, the two classes actually coincide in many cases. The first case is that when $C_c(\mathcal{A}/\Gamma)$ has an identity element:

Proposition 6.3.2. *If the crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ has an identity element (equivalently, if $C_c(\mathcal{A}/\Gamma)$ has an identity element), then every covariant pre-*-representation is a covariant *-representation.*

Proof: Let us assume that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ has an identity element (equivalently, $C_c(\mathcal{A}/\Gamma)$ has an identity element).

Let (π, ν) be a covariant pre-*-representation. As it was shown in Theorem 6.2.15, the integrated form $\pi \times \nu$ is a *-representation of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ such that $\nu = \mu_{\pi\times\nu}$, where $\mu_{\pi\times\nu}$ is the pre-*-representation which is obtained by extending $\pi \times \nu$ to the multiplier algebra $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$ and then restricting it to $\mathcal{H}(G, \Gamma)$. Since the crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ has an identity element, we have

$$M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) = C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma\,,$$

and therefore $\mu_{\pi\times\nu}$ is just the restriction of $\pi \times \nu$ to the the Hecke algebra $\mathcal{H}(G, \Gamma)$. Hence, $\nu = \mu_{\pi\times\nu}$ is a true *-representation. □

Another interesting situation where covariant pre-*-representations coincide with covariant *-representations is when $\mathcal{H}(G, \Gamma)$ is a BG^*-algebra. This is known to be the case for many classes of Hecke pairs (G, Γ), as we saw in Chapter 2.

Proposition 6.3.3. *If $\mathcal{H}(G, \Gamma)$ is a BG^*-algebra, then every covariant pre-*-representation is a covariant *-representation.*

Proof: If $\mathcal{H}(G, \Gamma)$ is a BG^*-algebra, then every pre-*-representation of $\mathcal{H}(G, \Gamma)$ is automatically normed and hence arises from a true *-representation. □

6.4 Crossed product in the case of free actions

In this section we will see that when the G-action on X is free the expressions for the products of the form $\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma$, described in Corollary 6.1.19, as well as the definition of a covariant pre-*-representation become much simpler and even more similar to the notion of *covariant pairs* of [19].

Theorem 6.4.1. *If the action of G on X is free, then*

$$\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} 1_{yu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{yv\Gamma} \tag{6.23}$$

and similarly,

$$\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} a_{xu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)v\Gamma}\,. \tag{6.24}$$

Lemma 6.4.2. *If the action of G on X is free, then*

$$n^y_{w,v} = 1 \qquad \textit{and} \qquad d^y_{w,v} = [\Gamma^{wv} : \Gamma^{wv} \cap w\Gamma w^{-1}]\,.$$

Proof: We have

$$\begin{aligned}
\mathfrak{n}^y_{w,v} &= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; yw^{-1} \in y\Gamma r^{-1}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; w^{-1} \in \Gamma r^{-1}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; r\Gamma = w\Gamma\big\} \\
&= \{w\Gamma\}\,.
\end{aligned}$$

Thus, $n^y_{w,v} = 1$. Also,

$$\begin{aligned}
\mathfrak{d}^y_{w,v} &= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; yw^{-1} \in y\Gamma r^{-1}\Gamma^{wv}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; w^{-1} \in \Gamma r^{-1}\Gamma^{wv}\big\}\,.
\end{aligned}$$

Now we notice that in the above set the condition $r^{-1}wv\Gamma \subseteq \Gamma v\Gamma$ is automatically satisfied from the second condition $w^{-1} \in \Gamma r^{-1}\Gamma^{wv}$, because the latter means that $r^{-1} = \theta_1 w^{-1}\theta_2$ for some $\theta_1 \in \Gamma$ and $\theta_2 \in \Gamma^{wv}$. Thus, we get

$$\begin{aligned}
\mathfrak{d}^y_{w,v} &= \big\{[r] \in \Gamma w\Gamma/\Gamma : w^{-1} \in \Gamma r^{-1}\Gamma^{wv}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r \in \Gamma^{wv}w\Gamma\big\} \\
&= \Gamma^{wv}w\Gamma/\Gamma\,.
\end{aligned}$$

Thus, we obtain $d^y_{w,v} = \#\big(\Gamma^{wv}w\Gamma/\Gamma\big) = [\Gamma^{wv} : \Gamma^{wv} \cap w\Gamma w^{-1}]$. $\square$

Proof of Theorem 6.4.1: We have seen in Theorem 6.1.17 that

$$\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \; \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g)N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right)$$

It follows from Lemma 6.4.2 that

$$N^{y\gamma}_{u^{-1},v} = \frac{1}{[\Gamma^{u^{-1}v} : \Gamma^{u^{-1}v} \cap u^{-1}\Gamma u]} .$$

Moreover, freeness of the action also implies that

$$\begin{aligned} E^y_{u,v} &= S_y \backslash \Gamma/(v\Gamma v^{-1} \cap u\Gamma u^{-1}) \\ &= \Gamma/(v\Gamma v^{-1} \cap u\Gamma u^{-1}) . \end{aligned}$$

Now, we have the following well-defined bijective correspondence

$$\begin{aligned} \Gamma/(\Gamma^u \cap \Gamma^v) &\longrightarrow \Gamma/(v\Gamma v^{-1} \cap u\Gamma u^{-1}) \\ [\theta] &\mapsto [\theta] , \end{aligned}$$

given by Proposition 1.2.1. Note that $\Gamma^u \cap \Gamma^v$ is simply the subgroup $u\Gamma u^{-1} \cap v\Gamma v^{-1} \cap \Gamma$, but in the following we will take preference on the notation $\Gamma^u \cap \Gamma^v$ for being shorter.

Consider now the action of Γ on $G/\Gamma \times G/\Gamma$ by left multiplication and denote by $\mathcal{O}_{h_1,h_2}$ the orbit of the element $(h_1\Gamma, h_2\Gamma) \in G/\Gamma \times G/\Gamma$. It is easy to see that the map

$$\begin{aligned} \Gamma/(\Gamma^{h_1} \cap \Gamma^{h_2}) &\longrightarrow \mathcal{O}_{h_1,h_2} \\ [\theta] &\mapsto (\theta h_1\Gamma, \theta h_2\Gamma) \end{aligned}$$

is also well-defined and is a bijection. We will denote by $\mathcal{C}$ the set of all orbits contained in $\Gamma g^{-1}\Gamma/\Gamma \times \Gamma s\Gamma/\Gamma$ (note that this set is Γ-invariant, so that it is a union of orbits). We then have

$$\begin{aligned} &\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g) N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right) \\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in\Gamma/(\Gamma^u\cap\Gamma^v)} \frac{\Delta(g) N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right) \\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in\Gamma/(\Gamma^u\cap\Gamma^v)} \frac{\Delta(g) N^{y}_{u^{-1}\gamma^{-1},\gamma v}}{L(u^{-1}\gamma^{-1}\gamma v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}\gamma^{-1}\gamma v\Gamma * 1_{y\gamma v\Gamma}\right) \end{aligned}$$

where the last equality comes from the fact that $N^{y\gamma}_{u^{-1},v} = N^{y}_{u^{-1}\gamma^{-1},\gamma v}$, which is a consequence of Lemma 6.1.18 *iii*), or simply by Lemma 6.4.2. Using now the bijection between $\Gamma/(\Gamma^u \cap \Gamma^v)$ and the orbit space $\mathcal{O}_{u,v}$ as described above, we obtain

$$\begin{aligned}
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{([\widetilde{u}],[\widetilde{v}])\in\mathcal{O}_{u,v}} \frac{\Delta(g)N^y_{\widetilde{u}^{-1},\widetilde{v}}}{L(\widetilde{u}^{-1}\widetilde{v})} \left(1_{y\widetilde{u}\Gamma} * \Gamma\widetilde{u}^{-1}\widetilde{v}\Gamma * 1_{y\widetilde{v}\Gamma}\right) \\
&= \sum_{\mathcal{O}\in\mathcal{C}} \sum_{([u],[v])\in\mathcal{O}} \sum_{([\widetilde{u}],[\widetilde{v}])\in\mathcal{O}_{u,v}} \frac{\Delta(g)N^y_{\widetilde{u}^{-1},\widetilde{v}}}{L(\widetilde{u}^{-1}\widetilde{v})} \left(1_{y\widetilde{u}\Gamma} * \Gamma\widetilde{u}^{-1}\widetilde{v}\Gamma * 1_{y\widetilde{v}\Gamma}\right) \\
&= \sum_{\mathcal{O}\in\mathcal{C}} \sum_{([u],[v])\in\mathcal{O}} \sum_{([\widetilde{u}],[\widetilde{v}])\in\mathcal{O}} \frac{\Delta(g)N^y_{\widetilde{u}^{-1},\widetilde{v}}an}{L(\widetilde{u}^{-1}\widetilde{v})} \left(1_{y\widetilde{u}\Gamma} * \Gamma\widetilde{u}^{-1}\widetilde{v}\Gamma * 1_{y\widetilde{v}\Gamma}\right) \\
&= \sum_{\mathcal{O}\in\mathcal{C}} \sum_{([\widetilde{u}],[\widetilde{v}])\in\mathcal{O}} \frac{\#\mathcal{O}\,\Delta(g)N^y_{\widetilde{u}^{-1},\widetilde{v}}}{L(\widetilde{u}^{-1}\widetilde{v})} \left(1_{y\widetilde{u}\Gamma} * \Gamma\widetilde{u}^{-1}\widetilde{v}\Gamma * 1_{y\widetilde{v}\Gamma}\right).
\end{aligned}$$

Changing the names of the variables ($\widetilde{u}$ to u and $\widetilde{v}$ to v) we get

$$\begin{aligned}
&= \sum_{\mathcal{O}\in\mathcal{C}} \sum_{([u],[v])\in\mathcal{O}} \frac{\#\mathcal{O}\,\Delta(g)N^y_{u^{-1},v}}{L(u^{-1}v)} \left(1_{yu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{yv\Gamma}\right) \\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\#\mathcal{O}_{u,v}\,\Delta(g)N^y_{u^{-1},v}}{L(u^{-1}v)} \left(1_{yu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{yv\Gamma}\right)
\end{aligned}$$

We are now going to prove that the coefficients satisfy

$$\frac{\#\mathcal{O}_{u,v}\,\Delta(g)N^y_{u^{-1},v}}{L(u^{-1}v)} = 1\,.$$

This follows from the following computation:

$$
\begin{aligned}
\frac{\#\mathcal{O}_{u,v}}{L(u^{-1}v)}\, N^{y}_{u^{-1},v}\, \Delta(g) &= \frac{[\Gamma:\Gamma^u\cap\Gamma^v]}{[\Gamma:\Gamma^{u^{-1}v}]}\cdot\frac{1}{[\Gamma^{u^{-1}v}:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]}\cdot\frac{[\Gamma:\Gamma^{u^{-1}}]}{[\Gamma:\Gamma^u]} \\
&= \frac{[\Gamma:\Gamma^u\cap\Gamma^v]\,[\Gamma:\Gamma^{u^{-1}}]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u][\Gamma:\Gamma^u]} \\
&= \frac{[\Gamma^u:\Gamma^u\cap\Gamma^v]\,[\Gamma:\Gamma^{u^{-1}}]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]} \\
&= \frac{[\Gamma^u:\Gamma^u\cap\Gamma^v]\,[u\Gamma u^{-1}:\Gamma^u]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]} \\
&= \frac{[u\Gamma u^{-1}:\Gamma^u\cap\Gamma^v]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]} \\
&= \frac{[u\Gamma u^{-1}:\Gamma^u\cap\Gamma^v]}{[u\Gamma u^{-1}:\Gamma^u\cap\Gamma^v]} \\
&= 1\,.
\end{aligned}
$$

This finishes the first claim of the theorem. The second claim, concerning the product $\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma$, is proven in a completely similar fashion. $\square$

Proposition 6.4.3. *Let $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation, $\mu : \mathcal{H}(G,\Gamma) \to L(\pi(C_c(\mathcal{A}/\Gamma)\mathscr{H})$ a pre-*-representation, let us assume that the G-action on X is free. The pair (π,μ) is a covariant pre-*-representation if and only if*

$$\pi(a_{x\Gamma})\mu(\Gamma g\Gamma) \in B(\pi(C_c(\mathcal{A}/\Gamma)\mathscr{H})\,, \tag{6.25}$$

for all $x \in X$, $a \in \mathcal{A}_x$ and $g \in G$, and if the following equality

$$\mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\mu(\Gamma s\Gamma) = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \pi(a_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)v\Gamma})\,. \tag{6.26}$$

holds in $L(\pi(C_c(\mathcal{A}/\Gamma)\mathscr{H})$ for all $g,s \in G$, $x \in X$ and $a \in \mathcal{A}_x$.

Proof: $(\Longrightarrow)$ Assume that (π,μ) is a covariant pre-*-representation.

Then (6.25) follows automatically and we have

$$\begin{aligned}\mu(\Gamma g\Gamma)\pi(a_{x\Gamma})\mu(\Gamma s\Gamma) &= [\pi\times\mu](\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma)\\ &= [\pi\times\mu]\Big(\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} a_{xu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)v\Gamma}\Big)\\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \pi(a_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)v\Gamma})\,.\end{aligned}$$

($\Longleftarrow$) Condition (6.18) is nothing but condition (6.25), whereas to prove equality (6.19) one just needs to show that

$$\begin{aligned}&\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\,\widetilde{\pi}(a_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})\\ =&\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \widetilde{\pi}(a_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)v\Gamma})\,,\end{aligned}$$

and this is proven in a completely analogous way as in the proof of Theorem 6.4.1. □

Chapter 7

Regular representations

In this chapter we introduce the notion of *regular representations* in the context of crossed products by Hecke pairs. These are concrete *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ involving the regular representation of the Hecke algebra $\mathcal{H}(G, \Gamma)$ and are indispensable for defining reduced C^*-crossed products.

The main novelty in the definition of a regular representation, as compared to the case of crossed products by groups, is that we have to take into account all algebras $C_c(\mathcal{A}/H)$, where $H = g_1 \Gamma g_1^{-1} \cap \cdots \cap g_n \Gamma g_n^{-1}$ is a finite intersection of conjugates of Γ. For this reason we need first to see how we can fit all these algebras together, which will be described in Section 7.1, before we can give the definition of a regular representation in Section 7.2.

7.1 Preliminaries

Proposition 7.1.1. *Suppose $K \subseteq H \subseteq G$ are subgroups of G, for which the action on X is H-good (hence also K-good). Suppose moreover that $[H : K] < \infty$. Then, there is an embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ determined by*

$$a_{xH} \longmapsto \sum_{[h] \in \mathcal{S}_x \backslash H/K} a_{xhK} \,.$$

Remark 7.1.2. We have already shown in Proposition 5.3.7 that inside the multiplier algebra $M(C_c(\mathcal{A}))$ the element a_{xH} decomposes as a sum of elements of $C_c(\mathcal{A}/K)$ as above. The point of Proposition 7.1.1 is that this decomposition really defines an embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$. Moreover, here we are not working inside $M(C_c(\mathcal{A}))$ anymore. Nevertheless this embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ is compatible with the embeddings of these algebras into $M(C_c(\mathcal{A}))$ as we will see at the end of this section.

Proof of Propostion 7.1.1: It is clear that the expression above is well-defined, since $[H:K]<\infty$, and it determines a linear map $\Phi: C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/K)$. It is also easy to see that this map is injective. The fact that Φ preserves the involution follows from the following computation

$$\begin{aligned}
\Phi((a_{xH})^*) &= \Phi(a^*_{x^{-1}H}) = \sum_{[h]\in\mathcal{S}_{x^{-1}}\backslash H/K} a^*_{x^{-1}hK} \\
&= \sum_{[h]\in\mathcal{S}_x\backslash H/K} a^*_{x^{-1}hK} = \Big(\sum_{[h]\in\mathcal{S}_x\backslash H/K} a_{xhK} \Big)^* \\
&= \Phi(a_{xH})^* .
\end{aligned}$$

Let us now check that Φ preserves products. If the pair (xH, yH) is not composable, then no pair of the form (xuK, ytK), with $u, t \in H$, is composable. Hence, in this case we have

$$\Phi(a_{xH}b_{yH}) = 0 = \Phi(a_{xH})\Phi(b_{yH}) .$$

Suppose now the pair (xH, yH) is composable, and let $\widetilde{h} \in H_{x,y}$. We have

$$\begin{aligned}
\Phi(a_{xH})\Phi(b_{yH}) &= \Big(\sum_{[u]\in\mathcal{S}_x\backslash H/K} a_{xuK}\Big)\Big(\sum_{[t]\in\mathcal{S}_y\backslash H/K} b_{ytK}\Big) \\
&= \sum_{[t]\in\mathcal{S}_y\backslash H/K}\ \sum_{[u]\in\mathcal{S}_x\backslash H/K} a_{xuK}b_{ytK} .
\end{aligned}$$

We now claim that $\sum_{[u]\in\mathcal{S}_x\backslash H/K} a_{xuK}b_{ytK} = (ab)_{(x\widetilde{h}t)(yt)K}$. To see this we notice that for $u = \widetilde{h}t$ we do have that the pair $(x\widetilde{h}tK, ytK)$ is composable and $a_{x\widetilde{h}tK}b_{ytK} = (ab)_{(x\widetilde{h}t)(yt)K}$. Now if $[u] \in \mathcal{S}_x\backslash H/K$ is such that the pair (xuK, ytK) is composable, then $\mathbf{s}(x)uK = \mathbf{r}(y)tK$. Since the pair $(x\widetilde{h}tK, ytK)$ is composable we also have $\mathbf{s}(x)\widetilde{h}tK = \mathbf{r}(y)tK$. Thus, $\mathbf{s}(x)uK = \mathbf{s}(x)\widetilde{h}tK$, i.e. $[u] = [\widetilde{h}t]$. This proves our claim and therefore we get

$$\begin{aligned}
\Phi(a_{xH})\Phi(b_{yH}) &= \sum_{[t]\in\mathcal{S}_y\backslash H/K}\ \sum_{[u]\in\mathcal{S}_x\backslash H/K} a_{xuK}b_{ytK} \\
&= \sum_{[t]\in\mathcal{S}_y\backslash H/K} (ab)_{(x\widetilde{h}t)(yt)K} \\
&= \sum_{[t]\in\mathcal{S}_y\backslash H/K} (ab)_{(x\widetilde{h}y)tK} .
\end{aligned}$$

Recall that since the G-action on X is H-good we have

$$\mathcal{S}_y \cap H = \mathcal{S}_{\mathbf{s}(y)} \cap H = \mathcal{S}_{x\widetilde{h}y} \cap H\,.$$

Therefore we get

$$\begin{aligned}\Phi(a_{xH})\Phi(b_{yH}) &= \sum_{[t]\in\mathcal{S}_{x\widetilde{h}y}\backslash H/K} (ab)_{(x\widetilde{h}y)tK} \\ &= \Phi((ab)_{x\widetilde{h}yH}) \\ &= \Phi(a_{xH}b_{yH})\,.\end{aligned}$$

Hence, Φ is an embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$. □

The canonical embeddings described in Proposition 7.1.1 are all compatible, as the following result shows:

Proposition 7.1.3. *Let $L \subseteq K \subseteq H$ be subgroups of G such that $[H : L] < \infty$ and for which the action is H-good (hence, K and L-good). The canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/L)$ factors through the canonical embeddings of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$, and $C_c(\mathcal{A}/K)$ into $C_c(\mathcal{A}/L)$. In other words, the following diagram of canonical embeddings commutes:*

$$\begin{tikzcd} C_c(\mathcal{A}/H) \arrow[r] \arrow[rr, bend right] & C_c(\mathcal{A}/K) \arrow[r] & C_c(\mathcal{A}/L) \end{tikzcd}$$

Proof: Let us denote by $\Phi_1 : C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/K)$, $\Phi_2 : C_c(\mathcal{A}/K) \to C_c(\mathcal{A}/L)$ and $\Phi_3 : C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/L)$ the canonical embeddings. We want to prove that $\Phi_3 = \Phi_2 \circ \Phi_1$. For this it is enough to check this equality on elements of the form a_{xH}. We have

$$\begin{aligned}\Phi_2 \circ \Phi_1(a_{xH}) &= \sum_{[h]\in\mathcal{S}_x\backslash H/K} \Phi_2(a_{xhK}) \\ &= \sum_{[h]\in\mathcal{S}_x\backslash H/K}\ \sum_{[k]\in\mathcal{S}_{xh}\backslash K/L} a_{xhkL}\,.\end{aligned}$$

We claim that if $h_1, \ldots, h_n \in H$ is a set or representatives of $\mathcal{S}_x\backslash H/K$, and if $k_1^i, \ldots, k_{r_i}^i$ is a set of representatives of $\mathcal{S}_{xh_i}\backslash K/L$ for each $i = 1, \ldots, n$, then the set of all products of the form $h_i k_j^i$ is a set of representatives for $\mathcal{S}_x\backslash H/L$. Let us start by proving that every two such products correspond to distinct elements of $\mathcal{S}_x\backslash H/L$. In other words, we want to show that if $[h_ik^i_j] = [h_lk^l_p]$ in $S_x\backslash H/L$, then $h_i = h_l$ and $k^i_j = k^l_p$. To see this we notice that the equality $[h_ik^i_j] = [h_lk^l_p]$ means that $xh_ik^i_jL = xh_lk^l_pL$, and therefore $xh_iK = xh_lK$, i.e. $[h_i] = [h_l]$ in $S_x\backslash H/K$, hence $h_i = h_l$ because these form a set a of representatives of $\mathcal{S}_x\backslash H/K$. Now, the equality $xh_ik^i_jL = xh_ik^i_pL$ means that $k^i_j = k^i_p$ for the same reasons. Now it remains to prove that any element of $[h] \in \mathcal{S}_x\backslash H/L$ has a representative of the form $h^ik^i_j$. To see this, first we take h_i such that $xhK = xh_iK$, and we consider an element $k \in K$ such that $xh = xh_ik$, obtaining $xhL = xh_ikL$. Now we take k^i_j such that $xh_ikL = xh_ik^i_jL$, and the result follows.

After proving the above claim we can now write

$$\begin{aligned}\Phi_2 \circ \Phi_1(a_{xH}) &= \sum_{[h]\in\mathcal{S}_x\backslash H/K}\ \sum_{[k]\in\mathcal{S}_{xh}\backslash K/L} a_{xhkL} \\ &= \sum_{[\widetilde{h}]\in\mathcal{S}_x\backslash H/L} a_{x\widetilde{h}L} \\ &= \Phi_3(a_{xH})\,.\end{aligned}$$

This finishes the proof. □

Let $\mathcal{C}$ be the set of all finite intersections of conjugates of Γ, i.e.

$$\mathcal{C} := \Big\{ \bigcap_{i=1}^{n} g_i\Gamma g_i^{-1} : n \in \mathbb{N}, g_1, \dots, g_n \in G \Big\}\,. \tag{7.1}$$

The set $\mathcal{C}$ becomes a directed set with respect to the partial order given by reverse inclusion of subgroups, i.e. $H_1 \leq H_2 \Leftrightarrow H_1 \supseteq H_2$, for any $H_1, H_2 \in \mathcal{C}$.

Since we are assuming that (G, Γ) is a Hecke-pair it is not difficult to see that for any $H_1, H_2 \in \mathcal{C}$ we have

$$H_1 \leq H_2 \implies [H_1 : H_2] < \infty\,.$$

Also, since we are assuming that the G-action on X is Γ-good and this property passes to conjugates and subgroups, it follows automatically that the action is also H-good, for any $H \in \mathcal{C}$.

The observations in the previous paragraph together with Proposition 7.1.3 imply that $\{C_c(\mathcal{A}/H)\}_{H\in\mathcal{C}}$ is a directed system of *-algebras. Let us denote by $D(\mathcal{A})$ the *-algebraic direct limit of this directed system, i.e.

$$D(\mathcal{A}) := \lim_{H\in\mathcal{C}} C_c(\mathcal{A}/H)\,.$$

There is an equivalent way of defining the algebra $D(\mathcal{A})$, by viewing it as the *-subalgebra of $M(C_c(\mathcal{A}))$ generated by all the $C_c(\mathcal{A}/H)$ with $H \in \mathcal{C}$, as we prove in the next result. This characterization of $D(\mathcal{A})$ is also a very useful one.

Proposition 7.1.4. *Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. Then the following diagram of canonical embeddings commutes:*

$$\begin{tikzcd} C_c(\mathcal{A}/H) \arrow[r] \arrow[dr] & C_c(\mathcal{A}/K) \arrow[d] \\ & M(C_c(\mathcal{A}))\,. \end{tikzcd} \tag{7.2}$$

As a consequence, $D(\mathcal{A})$ is *-isomorphic to the *-subalgebra of $M(C_c(\mathcal{A}))$ generated by all the $C_c(\mathcal{A}/H)$ with $H \in \mathcal{C}$.

Proof: We have to show that, inside $M(C_c(\mathcal{A}))$, we have

$$a_{xH} = \sum_{[h] \in \mathcal{S}_x \backslash H/K} a_{xhK}\,,$$

for all $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$. This was proven in Proposition 5.3.7.

Commutativity of the diagram (7.2) then implies that there exists a *-homomorphism from $D(\mathcal{A})$ to $M(C_c(\mathcal{A}))$ whose image is precisely the *-subalgebra generated by all $C_c(\mathcal{A}/H)$, with $H \in \mathcal{C}$. This *-homomorphism is injective since all the maps in the diagram (7.2) are injective. $\square$

7.2 Regular representations

In the theory of crossed products by groups $A \times G$, regular representations are the integrated forms of certain covariant representations involving the regular representation of G. They are defined in the following way: one starts with a nondegenerate representation π of A on some Hilbert space $\mathscr{H}$ and constructs a new representation π_α of A on the Hilbert $\mathscr{H} \otimes \ell^2(G)$, defined in an appropriate way, such that π_α together with the regular representation of G form a covariant representation. Their integrated form is then called a *regular representation.*

We are now going to make an analogous construction in the case of Hecke pairs. The main novelty here is that we have to start with a representation π

of $\mathcal{D}(\mathcal{A})$, instead of $C_c(\mathcal{A}/\Gamma)$, so that we can construct the new representation π_α of $C_c(\mathcal{A}/\Gamma)$. This is because we need to take into account all algebras of the form $C_c(\mathcal{A}/H)$, where $H = g_1\Gamma g_1^{-1} \cap \cdots \cap g_n\Gamma g_n^{-1}$ is a finite intersection of conjugates of Γ. Naturally, when Γ is a normal subgroup, $\mathcal{D}(\mathcal{A})$ is nothing but the algebra $C_c(\mathcal{A}/\Gamma)$ itself, so that we will recover the original definition of a regular representation for crossed products by groups.

Definition 7.2.1. Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. We define the map $\pi_\alpha : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H} \otimes \ell^2(G/\Gamma))$ by

$$\pi_\alpha(f)\ (\xi \otimes \delta_{h\Gamma}) := \pi(\alpha_h(f))\xi \otimes \delta_{h\Gamma}\,.$$

Proposition 7.2.2. *Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. Then, the map π_α is a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$.*

Lemma 7.2.3. *Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. Then the restriction of π to $C_c(\mathcal{A}/H)$ is nondegenerate, for any $H \in \mathcal{C}$.*

Proof: Let $\xi \in \mathscr{H}$ be such that $\pi(C_c(\mathcal{A}/H))\xi = 0$. Take any $x \in X$, $a \in \mathcal{A}_x$ and $K \in \mathcal{C}$ such that $K \subseteq H$. We have that

$$\begin{aligned} \|\pi(a_{xK})\xi\|^2 &= \langle \pi((a^*a)_{\mathbf{s}(x)K})\xi\,,\,\xi\rangle \\ &= \langle \pi(a^*_{x^{-1}K} \cdot a_{xH})\xi\,,\,\xi\rangle \\ &= \langle \pi(a^*_{x^{-1}K})\pi(a_{xH})\xi\,,\,\xi\rangle \\ &= 0\,. \end{aligned}$$

From this we conclude that $\pi(C_c(\mathcal{A}/K))\xi = 0$, for any $K \in \mathcal{C}$ such that $K \subseteq H$. Since for any subgroup $L \in \mathcal{C}$ we have $C_c(\mathcal{A}/L) \subseteq C_c(\mathcal{A}/(L \cap H))$, and obviously $L \cap H \subseteq H$, we can in fact conclude that $\pi(C_c(\mathcal{A}/L))\xi = 0$ for all $L \in \mathcal{C}$. In other words, we have proven that $\pi(D(\mathcal{A}))\xi = 0$, which by nondegeneracy of π implies that $\xi = 0$. $\square$

Proof of Proposition 7.2.2: It is clear that the expression that defines $\pi_\alpha(f)$, for $f \in C_c(\mathcal{A}/\Gamma)$, defines a linear operator on the inner product space $\mathscr{H} \otimes C_c(G/\Gamma)$. Let us first check that this operator is indeed bounded. We have

$$\begin{aligned}
\|\pi_\alpha(f)\Big(\sum_{[h]\in G/\Gamma}\xi_{h\Gamma}\otimes\delta_{h\Gamma}\Big)\|^2 &= \|\sum_{[h]\in G/\Gamma}\pi(\alpha_h(f))\xi_{h\Gamma}\otimes\delta_{h\Gamma}\|^2 \\
&= \sum_{[h]\in G/\Gamma}\|\pi(\alpha_h(f))\xi_{h\Gamma}\|^2 \\
&\leq \sum_{[h]\in G/\Gamma}\|\pi(\alpha_h(f))\|^2\|\xi_{h\Gamma}\|^2 \\
&\leq \sum_{[h]\in G/\Gamma}\|\alpha_h(f)\|^2_{C^*(\mathcal{A}/h\Gamma h^{-1})}\|\xi_{h\Gamma}\|^2 .
\end{aligned}$$

Since α_h gives an isomorphism between $C^*(\mathcal{A}/\Gamma)$ and $C^*(\mathcal{A}/h\Gamma h^{-1})$ we get

$$\begin{aligned}
&= \sum_{[h]\in G/\Gamma}\|f\|^2_{C^*(\mathcal{A}/\Gamma)}\|\xi_{h\Gamma}\|^2 \\
&= \|f\|^2_{C^*(\mathcal{A}/\Gamma)}\|\sum_{[h]\in G/\Gamma}\xi_{h\Gamma}\otimes\delta_{h\Gamma}\|^2 .
\end{aligned}$$

Hence, $\pi_\alpha(f)$ is bounded and thus defines uniquely an operator in $B(\mathscr{H}\otimes\ell^2(G/\Gamma))$. It is simple to check that π is linear and preserves products. Let us then see that it preserves the involution. We have

$$\begin{aligned}
\langle\pi_\alpha(f)\,(\xi\otimes\delta_{h\Gamma})\,,\,\eta\otimes\delta_{g\Gamma}\rangle &= \langle\pi(\alpha_h(f))\xi\otimes\delta_{h\Gamma}\,,\,\eta\otimes\delta_{g\Gamma}\rangle \\
&= \langle\pi(\alpha_h(f))\xi\,,\,\eta\rangle\langle\delta_{h\Gamma}\,,\,\delta_{g\Gamma}\rangle \\
&= \langle\xi\,,\,\pi(\alpha_h(f^*))\eta\rangle\langle\delta_{h\Gamma}\,,\,\delta_{g\Gamma}\rangle \\
&= \langle\xi\,,\,\pi(\alpha_g(f^*))\eta\rangle\langle\delta_{h\Gamma}\,,\,\delta_{g\Gamma}\rangle \\
&= \langle\xi\otimes\delta_{h\Gamma}\,,\,\pi_\alpha(f^*)\,(\eta\otimes\delta_{g\Gamma})\rangle .
\end{aligned}$$

Thus, $\pi_\alpha(f)^* = \pi_\alpha(f^*)$, and therefore π_α defines a *-representation. It remains to check that this *-representation is nondegenerate. To see this, we start by canonically identifying $\mathscr{H}\otimes\ell^2(G/\Gamma)$ with the Hilbert space $\ell^2(G/\Gamma,\mathscr{H})$. On this Hilbert space, it is easy to see that $\pi_\alpha(f)$ is given by

$$[\pi_\alpha(f)\,(\zeta)]\,(h\Gamma) = \pi(\alpha_h(f))\,\zeta(h\Gamma)$$

for $\zeta\in\ell^2(G/\Gamma,\mathscr{H})$. Suppose now that $\zeta\in\ell^2(G/\Gamma,\mathscr{H})$ is such that $\pi_\alpha(f)\,\zeta = 0$ for all $f\in C_c(\mathcal{A}/\Gamma)$. Thus, for each $h\Gamma\in G/\Gamma$ we have $\pi(\alpha_h(f))\,\zeta(h\Gamma) = 0$ for all $f\in C_c(\mathcal{A}/\Gamma)$. This can be expressed equivalently as $\pi(f)\,\zeta(h\Gamma) = 0$ for all $f\in C_c(\mathcal{A}/h\Gamma h^{-1})$. By Lemma 7.2.3 the restriction of π to $C_c(\mathcal{A}/h\Gamma h^{-1})$ is nondegenerate and therefore we have $\zeta(h\Gamma) = 0$. Thus, π_α is nondegenerate. $\square$

Definition 7.2.4. Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation and $\rho : \mathcal{H}(G,\Gamma) \to B(\ell^2(G/\Gamma))$ the right regular representation of the Hecke algebra. The pair $(\pi_\alpha, 1\otimes\rho)$ is called a *regular covariant representation.*

Remark 7.2.5. We observe that when Γ is a normal subgroup of G we have $g\Gamma g^{-1} = \Gamma$ for all $g \in G$, so that the algebra $D(\mathcal{A})$ coincides with $C_c(\mathcal{A}/\Gamma)$. For this reason our notion of a regular representation coincides with the usual notion of a regular covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma, \alpha)$.

Theorem 7.2.6. *Every regular covariant representation $(\pi_\alpha, 1\otimes\rho)$ is a covariant *-representation. Moreover, its integrated form is given by*

$$[\pi_\alpha \times (1\otimes\rho)](f)\;(\xi\otimes\delta_{h\Gamma}) = \sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\xi\otimes\delta_{g\Gamma}\,, \tag{7.3}$$

for every $f \in C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$.

Proof: We shall first check that the expression (7.3) does indeed define a *-representation of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$. Afterwards we will show that the covariant pre-*-representation associated to it is precisely $(\pi_\alpha, 1\otimes\rho)$.

Let $\pi_{reg} : C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma \to B(\mathscr{H}\otimes\ell^2(G/\Gamma))$ be defined by

$$\pi_{reg}(f)\;(\xi\otimes\delta_{h\Gamma}) := \sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\xi\otimes\delta_{g\Gamma}\,.$$

It is not evident that π_{reg} is a bounded operator for all $f \in C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$, but it is clear that $\pi_{reg}(f)$ is well-defined as a linear operator on the inner product space $\mathscr{H}\otimes C_c(G/\Gamma)$. Under the identification of $\mathscr{H}\otimes C_c(G/\Gamma)$ with $C_c(G/\Gamma, \mathscr{H})$, it is easy to see that $\pi_{reg}(f)$ is given by

$$[\pi_{reg}(f)\,\eta]\,(g\Gamma) = \sum_{[h]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\eta(h\Gamma)\,,$$

for any $\eta \in C_c(G/\Gamma, \mathscr{H})$. Let us now check that $\pi_{reg}(f)$ is indeed bounded. For any vector $\eta \in C_c(G/\Gamma, \mathscr{H})$ we have

$$\begin{aligned}
\|\pi_{reg}(f)\,\eta\|^2 &= \sum_{[g]\in G/\Gamma} \|[\pi_{reg}(f)\,\eta](g\Gamma)\|^2\\
&= \sum_{[g]\in G/\Gamma} \Big\| \sum_{[h]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\eta(h\Gamma)\;\Big\|^2\\
&\leq \sum_{[g]\in G/\Gamma} \Big(\sum_{[h]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\|\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\|\|\eta(h\Gamma)\|\Big)^2 .
\end{aligned}$$

For each $h\Gamma \in G/\Gamma$ let us define $T^{h\Gamma} \in C_c(G/\Gamma)$ by

$$T^{h\Gamma}(g\Gamma) := \Delta(g^{-1}h)^{\frac{1}{2}} \, \|\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\| \|\eta(h\Gamma)\| \,,$$

and $T \in C_c(G/\Gamma)$ by $T := \sum_{[h]\in G/\Gamma} T^{h\Gamma}$, which is clearly a finite sum since η has finite support. Thus, we have

$$\begin{aligned}
\|\pi_{reg}(f)\,\eta\|^2 &\leq \sum_{[g]\in G/\Gamma} \Big(\sum_{[h]\in G/\Gamma} T^{h\Gamma}(g\Gamma)\Big)^2 \\
&= \sum_{[g]\in G/\Gamma} (T(g\Gamma))^2 \\
&= \|T\|^2_{\ell^2(G/\Gamma)} \\
&= \| \sum_{[h]\in G/\Gamma} T^{h\Gamma} \|^2_{\ell^2(G/\Gamma)} \\
&\leq \Big(\sum_{[h]\in G/\Gamma} \|T^{h\Gamma}\|_{\ell^2(G/\Gamma)}\Big)^2 \\
&= \Bigg(\sum_{[h]\in G/\Gamma} \sqrt{\sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)\, \|\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\|^2 \|\eta(h\Gamma)\|^2} \,\Bigg)^2 \\
&= \Bigg(\sum_{[h]\in G/\Gamma} \|\eta(h\Gamma)\| \sqrt{\sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)\, \|\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\|^2} \,\Bigg)^2 .
\end{aligned}$$

By the Cauchy-Schwarz inequality in $\ell^2(G/\Gamma)$ we get

$$\begin{aligned}
&\leq \Big(\sum_{[h]\in G/\Gamma} \|\eta(h\Gamma)\|^2\Big) \Big(\sum_{[h]\in G/\Gamma} \sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)\, \|\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\|^2 \Big) \\
&= \Big(\sum_{[h]\in G/\Gamma} \sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \|\pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\|^2 \Big) \|\eta\|^2 \,,
\end{aligned}$$

which shows that $\pi_{reg}(f)$ is bounded.

Let us now check that π_{reg} preserves products and the involution. Let $f_1, f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$. We have

$$
\begin{aligned}
&\pi_{reg}(f_1 * f_2)\left(\xi \otimes \delta_{h\Gamma}\right) = \\
= \;& \sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g\big((f_1 * f_2)(g^{-1}h\Gamma)\big)\big)\xi \otimes \delta_{g\Gamma} \\
= \;& \sum_{[g]\in G/\Gamma} \sum_{[s]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g\big((f_1(s\Gamma)\alpha_s(f_2(s^{-1}g^{-1}h\Gamma))\big)\big)\xi \otimes \delta_{g\Gamma} \\
= \;& \sum_{[g]\in G/\Gamma} \sum_{[s]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g(f_1(s\Gamma))\alpha_{gs}(f_2(s^{-1}g^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \\
= \;& \sum_{[g]\in G/\Gamma} \sum_{[s]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g(f_1(g^{-1}s\Gamma))\alpha_s(f_2(s^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \\
= \;& \sum_{[s]\in G/\Gamma} \sum_{[g]\in G/\Gamma} \Delta(g^{-1}s)^{\frac{1}{2}}\, \Delta(s^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g(f_1(g^{-1}s\Gamma))\big)\pi\big(\alpha_s(f_2(s^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \\
= \;& \sum_{[s]\in G/\Gamma} \pi_{reg}(f_1)\left(\Delta(s^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_s(f_2(s^{-1}h\Gamma))\big)\xi \otimes \delta_{s\Gamma}\right) \\
= \;& \pi_{reg}(f_1)\pi_{reg}(f_2)\left(\xi \otimes \delta_{h\Gamma}\right).
\end{aligned}
$$

Hence we conclude that $\pi_{reg}(f_1 * f_2) = \pi_{reg}(f_1)\pi_{reg}(f_2)$. Let us now check that π_{reg} preserves the involution. For $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ we have

$$
\begin{aligned}
&\left\langle \pi_{reg}(f^*)\left(\xi \otimes \delta_{h\Gamma}\right), \eta \otimes \delta_{s\Gamma} \right\rangle = \\
= \;& \sum_{[g]\in G/\Gamma} \left\langle \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g(f^*(g^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma}, \eta \otimes \delta_{s\Gamma} \right\rangle \\
= \;& \sum_{[g]\in G/\Gamma} \left\langle \Delta(g^{-1}h)^{\frac{1}{2}} \Delta(h^{-1}g)\, \pi\big(\alpha_g(\alpha_{g^{-1}h}(f(h^{-1}g\Gamma))^*)\big)\xi \otimes \delta_{g\Gamma}, \eta \otimes \delta_{s\Gamma} \right\rangle \\
= \;& \sum_{[g]\in G/\Gamma} \left\langle \Delta(h^{-1}g)^{\frac{1}{2}}\, \pi\big(\alpha_h(f(h^{-1}g\Gamma))\big)^*\xi, \eta \right\rangle \left\langle \delta_{g\Gamma}, \delta_{s\Gamma} \right\rangle \\
= \;& \left\langle \xi, \Delta(h^{-1}s)^{\frac{1}{2}}\, \pi\big(\alpha_h(f(h^{-1}s\Gamma))\big)\eta \right\rangle.
\end{aligned}
$$

On the other side we also have

$$
\begin{aligned}
&\left\langle \xi \otimes \delta_{h\Gamma}, \pi_{reg}(f)\left(\eta \otimes \delta_{s\Gamma}\right) \right\rangle = \\
= \;& \sum_{[g]\in G/\Gamma} \left\langle \xi \otimes \delta_{h\Gamma}, \Delta(g^{-1}s)^{\frac{1}{2}}\, \pi\big(\alpha_g(f(g^{-1}s\Gamma))\big)\eta \otimes \delta_{g\Gamma} \right\rangle \\
= \;& \sum_{[g]\in G/\Gamma} \left\langle \xi, \Delta(g^{-1}s)^{\frac{1}{2}}\, \pi\big(\alpha_g(f(g^{-1}s\Gamma))\big)\eta \right\rangle \left\langle \delta_{h\Gamma}, \delta_{g\Gamma} \right\rangle \\
= \;& \left\langle \xi, \Delta(h^{-1}s)^{\frac{1}{2}}\, \pi\big(\alpha_h(f(h^{-1}s\Gamma))\big)\eta \right\rangle.
\end{aligned}
$$

Therefore we can conclude that $\pi_{reg}(f^*) = \pi_{reg}(f)^*$. Hence, π_{reg} is a *-representation.

The restriction of π_{reg} to $C_c(\mathcal{A}/\Gamma)$ is precisely π_α, and since π_α is nondegenerate, then so is π_{reg}. Hence, it follows from Theorem 6.2.15 that π_{reg} is the integrated form of a covariant pre-*-representation $(\pi_{reg}|, \mu_{\pi_{reg}})$, as defined in Proposition 6.2.10. As we pointed out above, $\pi_{reg}| = \pi_\alpha$. Thus, to finish the proof we only need to prove that $\mu_{\pi_{reg}} = 1 \otimes \rho$. For a vector of the form $\pi_\alpha(a_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) \in \pi_\alpha(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ and a double coset $\Gamma g \Gamma$ we have

$$\begin{aligned}
\mu_{\pi_{reg}}(\Gamma g\Gamma)\,\pi_\alpha(a_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) &= \widetilde{\pi_{reg}}(\Gamma g\Gamma)\,\pi_\alpha(a_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) \\
&= \widetilde{\pi_{reg}}(\Gamma g\Gamma)\pi_{reg}(a_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) \\
&= \pi_{reg}(\Gamma g\Gamma * a_{x\Gamma})(\xi \otimes \delta_{h\Gamma})\,.
\end{aligned}$$

Let us now compute $\pi_{reg}(f)(\xi \otimes \delta_{h\Gamma})$ for $f := \Gamma g\Gamma * a_{x\Gamma}$. By definition

$$\pi_{reg}(f)\;(\xi \otimes \delta_{h\Gamma}) = \sum_{[s]\in G/\Gamma} \Delta(s^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_s(f(s^{-1}h\Gamma))\big)\xi \otimes \delta_{s\Gamma}\,.$$

It is clear that $f(s^{-1}h\Gamma)$ is nonzero if and only if $s^{-1}h\Gamma \subseteq \Gamma g\Gamma$, which is equivalent to $s\Gamma \subseteq h\Gamma g^{-1}\Gamma$. Hence,

$$\begin{aligned}
&= \sum_{[s]\in\, h\Gamma g^{-1}\Gamma/\Gamma} \Delta(s^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_s(f(s^{-1}h\Gamma))\big)\xi \otimes \delta_{s\Gamma} \\
&= \sum_{[\theta]\in\Gamma/\Gamma^{g^{-1}}} \Delta(g\theta^{-1}h^{-1}h)^{\frac{1}{2}}\,\pi\big(\alpha_{h\theta g^{-1}}(f(g\theta^{-1}h^{-1}h\Gamma))\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma} \\
&= \sum_{[\theta]\in\Gamma/\Gamma^{g^{-1}}} \Delta(g)^{\frac{1}{2}}\,\pi\big(\alpha_{h\theta g^{-1}}(f(g\Gamma))\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma}\,.
\end{aligned}$$

Now, it is easily seen that $f(g\Gamma) = a_{xg^{-1}g\Gamma g^{-1}}$. Hence, we get

$$\begin{aligned}
&= \sum_{[\theta]\in\Gamma/\Gamma^{g^{-1}}} \Delta(g)^{\frac{1}{2}}\,\pi\big(\alpha_{h\theta g^{-1}}(a_{xg^{-1}g\Gamma g^{-1}})\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma} \\
&= \sum_{[\theta]\in\Gamma/\Gamma^{g^{-1}}} \Delta(g)^{\frac{1}{2}}\,\pi\big(\alpha_h(a_{x\Gamma})\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma} \\
&= (1 \otimes \rho)(\Gamma g\Gamma)\,\Big(\pi\big(\alpha_h(a_{x\Gamma})\big)\xi \otimes \delta_{h\Gamma}\Big) \\
&= (1 \otimes \rho)(\Gamma g\Gamma)\,\pi_\alpha(a_{x\Gamma})\,\big(\xi \otimes \delta_{h\Gamma}\big)\,.
\end{aligned}$$

This shows that $\mu_{\pi_{reg}} = 1 \otimes \rho$ in $\pi_\alpha(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ and finishes the proof.

□

Chapter 8

Reduced C^*-crossed products

In this chapter we define reduced C^*-crossed products by Hecke pairs and study some of their properties. Since the algebra $C_c(\mathcal{A}/\Gamma)$ admits several C^*-completions, we will be able to form several reduced C^*-crossed products, such as $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. As we shall see, many of the main properties of reduced C^*-crossed products by groups hold also in the Hecke pair case.

In Section 8.4 we also compare our construction of a reduced crossed product by a Hecke pair with that of Laca, Larsen and Neshveyev in [30], and show that they agree whenever they are both definable.

8.1 Preliminaries

There are two reduced C^*-crossed products by Hecke pairs which are of particular interest to us, and these are $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. These will be defined and studied, in a singular approach, in section 8.2, but for that we need first to understand how the canonical embeddings

$$C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/K)\,, \tag{8.1}$$

defined in Proposition 7.1.1 for $K \subseteq H$ such that $[H : K] < \infty$, behave with respect the full and reduced C^*-completions. The goal of this preliminary section is exactly to show that this embedding always gives rise to embeddings in the two canonical C^*-completions

$$C^*_r(\mathcal{A}/H) \to C^*_r(\mathcal{A}/K) \qquad \text{and} \qquad C^*(\mathcal{A}/H) \to C^*(\mathcal{A}/K)\,.$$

8.1.1 Reduced completions $C_r^*(\mathcal{A}/H)$

The purpose of this subsection is to prove the following result:

Theorem 8.1.1. *Let $K \subseteq H \subseteq G$ be subgroups such that $[H:K] < \infty$. The canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ completes to an embedding of $C_r^*(\mathcal{A}/H)$ into $C_r^*(\mathcal{A}/K)$.*

In order to prove this result we need to establish some notation and some lemmas first. Even though Theorem 8.1.1 is stated for subgroups $K \subseteq H$ for which we have a finite index $[H:K]$ we will state and prove the two following lemmas in greater generality, as it will be convenient later on.

Recall, from Proposition 5.3.6, that for any two subgroups $K \subseteq H$ of G for which the G-action is H-good we have that, inside $M(C_c(\mathcal{A}))$, the algebra $C_c(\mathcal{A}/H)$ acts on $C_c(\mathcal{A}/K)$ in the following way:

$$a_{xH} b_{yK} = \begin{cases} (ab)_{x\widetilde{h}yK}\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

where $\widetilde{h}$ is any element of $H_{x,y}$. As a consequence, this action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A}/K)$ defines a *-homomorphism

$$C_c(\mathcal{A}/H) \to M(C_c(\mathcal{A}/K))\,.$$

It could be proven (in the same fashion as Theorem 5.3.1) that the *-homomorphism above is in fact an embedding, but we will not need this fact here.

We make the following definition:

Definition 8.1.2. Suppose A is *-algebra and B is a C^*-algebra. A *right $A-B$ bimodule* X is a (right) inner product B-module (in the sense of [37, Definition 2.1]) which is also a left A-module satisfying:

$$a(xb) = (ax)b\,,$$
$$\langle ax, y\rangle_B = \langle x, a^*y\rangle_B\,,$$

for all $x, y \in X$, $a \in A$ and $b \in B$.

Given a right $A-B$ bimodule X we will say that A *acts by bounded operators* on X if for any $a \in A$ there exists $C > 0$ such that

$$\|ax\|_B \leq C\|x\|_B\,,$$

for every $x \in X$, where $\|\cdot\|_B$ is the norm induced by $\langle \cdot\,,\,\cdot\rangle_B$.

If A is a *-algebra which has an enveloping C^*-algebra $C^*(A)$, then any right $A - B$ bimodule where A acts by bounded operators can be completed to a right-Hilbert $C^*(A) - B$ bimodule.

Lemma 8.1.3. *Let $K \subseteq H$ be subgroups of G and let D be a C^*-algebra. Suppose $C_c(\mathcal{A}/K)$ is an inner product D-module, denoted by $C_c(\mathcal{A}/K)_D$. Assume furthermore that $C_c(\mathcal{A}/K)_D$ is a right $C_c(\mathcal{A}/K) - D$ bimodule and also a right $C_c(\mathcal{A}/H) - D$ bimodule, where $C_c(\mathcal{A}/K)$ acts on itself by right multiplication and $C_c(\mathcal{A}/H)$ acts on $C_c(\mathcal{A}/K)$ in the canonical way.*

If $C_c(\mathcal{A}/K)$ acts on $C_c(\mathcal{A}/K)_D$ by bounded operators, then $C_c(\mathcal{A}/H)$ also acts on $C_c(\mathcal{A}/K)_D$ by bounded operators.

Proof: Suppose that $C_c(\mathcal{A}/K)$ acts on $C_c(\mathcal{A}/K)_D$ by bounded operators. We need to show that $C_c(\mathcal{A}/H)$ also acts on $C_c(\mathcal{A}/K)_D$ by bounded operators, with respect to the norm $\|\cdot\|_D$. For this it is enough to prove that the maps

$$a_{xH} : C_c(\mathcal{A}/K) \to C_c(\mathcal{A}/K)\,,$$

are bounded with respect to the norm $\|\cdot\|_D$. Moreover, from the fact that $(a_{xH})^* a_{xH} = (a^*a)_{s(x)H}$ it actually suffices to show that any mapping $a_{uH} : C_c(\mathcal{A}/K) \to C_c(\mathcal{A}/K)$ is bounded relatively to the norm $\|\cdot\|_D$, for any unit $u \in X^0$.

As we have seen before, we can write any element $f \in C_c(\mathcal{A}/K)$ as a sum of the form $\sum_{yK \in X/K} (f(yK))_{yK}$. Furthermore, we can split the sum according the ranges of elements, i.e.

$$f = \sum_{yK \in X/K} (f(yK))_{yK} = \sum_{vK \in X^0/K} \sum_{\substack{yK \in X/K \\ \mathbf{r}(y)K = vK}} (f(yK))_{yK}\,.$$

Applying the multiplier a_{uH} to this element we get

$$\begin{aligned} a_{uH} f &= a_{uH} \sum_{vK \in X^0/K} \sum_{\substack{yK \in X/K \\ \mathbf{r}(y)K = vK}} (f(yK))_{yK} \\ &= \sum_{vK \subseteq uH} \sum_{\substack{yK \in X/K \\ \mathbf{r}(y)K = vK}} (af(yK))_{yK} \\ &= \sum_{vK \subseteq uH} \sum_{\substack{yK \in X/K \\ \mathbf{r}(y)K = vK}} a_{vK} (f(yK))_{yK}\,. \end{aligned}$$

Since f has compact support, there is necessarily a finite number of elements $v_1K, \ldots, v_nK \subseteq uH$ such that

$$\begin{aligned}
a_{uH} f &= \sum_{i=1}^{n} \sum_{\substack{yK \in X/K \\ \mathbf{r}(y)K = v_iK}} a_{v_iK}(f(yK))_{yK} \\
&= \big(\sum_{i=1}^{n} a_{v_iK}\big)\big(\sum_{yK \in X/K} (f(yK))_{yK}\big) \\
&= \big(\sum_{i=1}^{n} a_{v_iK}\big)\, f\,.
\end{aligned}$$

Our assumptions say that left multiplication by elements of $C_c(\mathcal{A}/K)$ is continuous with respect to $\|\cdot\|_D$. Denoting by $\overline{C_c(\mathcal{A}/K)_D}$ the completion of $C_c(\mathcal{A}/K)_D$ as a Hilbert D-module, we have that every element of $C_c(\mathcal{A}/K)$ uniquely defines an element of $\mathcal{L}\big(\overline{C_c(\mathcal{A})_D}\big)$. Denoting by $\|\cdot\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A})_D}\right)}$ the operator norm in $\mathcal{L}\big(\overline{C_c(\mathcal{A})_D}\big)$, we have

$$\begin{aligned}
\|a_{uH} f\,\|_D &= \|\big(\sum_{i=1}^{n} a_{v_iK}\big) f\|_D \\
&\leq \|\sum_{i=1}^{n} a_{v_i}\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A})_D}\right)} \; \|f\|_D\,,
\end{aligned}$$

Now we notice that $\sum_{i=1}^{n} a_{v_iK}$ is an element of a bundle of C^*-algebras over the units $v_1K, \ldots, v_nK$, and therefore we must have, by uniqueness of C^*-norms on C^*-algebras,

$$\|\sum_{i=1}^{n} a_{v_iK}\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A})_D}\right)} = \max_i \|a_{v_iK}\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A})_D}\right)} = \|a\|\,.$$

Hence we conclude that $\|a_{uH} f\|_D \leq \|a\|\,\|f\|_D$, i.e. a_{uH} is bounded. $\square$

Let us now consider $C_c(\mathcal{A}/K)$ as the right $C_c(\mathcal{A}/K)$–$C_0(\mathcal{A}^0/K)$ bimodule whose completion is the right-Hilbert bimodule ${}_{C^*(\mathcal{A}/K)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$. We claim that the canonical action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A}/K)$ makes $C_c(\mathcal{A}/K)$ into a right $C_c(\mathcal{A}/H) - C_0(\mathcal{A}/K)$ bimodule. The fact that $f_1(\xi f_2) = (f_1\xi)f_2$, for any $f_1 \in C_c(\mathcal{A}/H)$, $\xi \in C_c(\mathcal{A}/K)$ and $f_2 \in C_0(\mathcal{A}^0/K)$, is obvious. Thus, we only need to check that $\langle f\xi\,,\,\eta\rangle_{C_0(\mathcal{A}^0/K)} = \langle \xi\,,\,f^*\eta\rangle_{C_0(\mathcal{A}^0/K)}$, for any $f \in$

$C_c(\mathcal{A}/H)$ and $\xi, \eta \in C_c(\mathcal{A}/K)$. This is also easy to see because, by definition,

$$\begin{aligned} \langle f\xi \,,\, \eta\rangle_{C_0(\mathcal{A}^0/K)} &= ((f\xi)^*\eta)|_{C_0(\mathcal{A}/K)} \\ &= (\xi^*(f^*\eta))|_{C_0(\mathcal{A}/K)} \\ &= \langle \xi \,,\, f^*\eta\rangle_{C_0(\mathcal{A}^0/K)} \,. \end{aligned}$$

Hence, we are under the conditions of Lemma 8.1.3, and therefore the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$ is by bounded operators. Hence, the right $C_c(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodule $C_c(\mathcal{A}/K)$ can be completed to a right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$.

Lemma 8.1.4. *The *-homomorphism $\Phi : C^*(\mathcal{A}/H) \to \mathcal{L}(L^2(\mathcal{A}/K))$ associated with the right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$ has the same kernel as the canonical map $\Lambda : C^*(\mathcal{A}/H) \to C^*_r(\mathcal{A}/H)$.*

Proof: The proof of this fact is essentially an adaptation of the proof of [10, Proposition 2.10], and is achieved by exhibiting two isomorphic right-Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodules Y and Z such that the *-homomorphisms of $C^*(\mathcal{A}/H)$ into $\mathcal{L}(Y)$ and $\mathcal{L}(Z)$ have the same kernels as Λ and Φ respectively.

We naturally have a right-Hilbert bimodule ${}_{C_0(\mathcal{A}^0/H)}C_0(\mathcal{A}^0/K)_{C_0(\mathcal{A}^0/K)}$, where the action of $C_0(\mathcal{A}^0/H)$ on $C_0(\mathcal{A}^0/K)$ extends the action of $C_c(\mathcal{A}^0/H)$ on $C_c(\mathcal{A}^0/K)$. We define Y as the balanced tensor product of the right-Hilbert bimodules ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/H)_{C_0(\mathcal{A}^0/H)}$ and ${}_{C_0(\mathcal{A}^0/H)}C_0(\mathcal{A}^0/K)_{C_0(\mathcal{A}^0)/K}$, i.e.

$$Y := L^2(\mathcal{A}/H) \otimes_{C_0(\mathcal{A}^0/H)} C_0(\mathcal{A}^0/K) \,.$$

Since $C_0(\mathcal{A}^0/H)$ acts faithfully on $C_0(\mathcal{A}^0/K)$, the associated *-homomorphism of $C^*(\mathcal{A}/H)$ to $\mathcal{L}(Y)$ has the same kernel as Λ. We define Z simply as

$${}_{C^*(\mathcal{A}/H)}Z_{C_0(\mathcal{A}^0/K)} := {}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)} \,.$$

We now want to define an isomorphism $\Psi : L^2(\mathcal{A}/H) \otimes_{C_0(\mathcal{A}^0/H)} C_0(\mathcal{A}^0/K) \to L^2(\mathcal{A}/K)$ of Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodules. We start by defining

$$\begin{aligned} \Psi_0 : C_c(\mathcal{A}/H) \otimes_{C_c(\mathcal{A}^0/H)} C_c(\mathcal{A}^0/K) &\longrightarrow L^2(\mathcal{A}/K) \,, \\ \Psi_0(f_1 \otimes f_2) &:= f_1 \cdot f_2 \,. \end{aligned}$$

It is easy to see that Ψ_0 is well-defined. To see that Ψ_0 preserves the inner products it is enough to check on the generators. So let $a_{xH}, b_{yH} \in C_c(\mathcal{A}/H)$

and $c_{uK}, d_{vK} \in C_c(\mathcal{A}^0/K)$, with $u, v \in X^0$. We have

$$\begin{aligned}
\langle \Psi_0(a_{xH} \otimes c_{uK})\,,\, \Psi_0(b_{yH} \otimes d_{vK}) \rangle_{C_0(\mathcal{A}^0/K)} &= \langle a_{xH}c_{uK}\,,\, b_{yH}d_{vK} \rangle_{C_0(\mathcal{A}^0/K)} \\
&= (c^*_{uK}a^*_{x^{-1}H}b_{yH}d_{vK})|_{C_0(\mathcal{A}^0/K)}\,.
\end{aligned}$$

Now the product $(c^*_{uK}a^*_{x^{-1}H}b_{yH}d_{vK})|_{C_0(\mathcal{A}^0/K)}$ is automatically zero unless $vK = uK$, $xH = yH$ and $vK \subseteq \mathbf{s}(y)H$, in which case we necessarily have that $(c^*_{uK}a^*_{x^{-1}H}b_{yH}d_{vK})|_{C_0(\mathcal{A}^0/K)} = (c^*a^*bd)_{vK}$. On the other hand,

$$\begin{aligned}
\langle a_{xH} \otimes c_{uK}\,,\, b_{yH} \otimes d_{vK} \rangle_{C_0(\mathcal{A}^0/K)} &= \langle c_{uK}\,,\, \langle a_{xH}\,,\, b_{yH} \rangle_{C_0(\mathcal{A}^0/H)}\, d_{vK} \rangle_{C_0(\mathcal{A}^0/K)} \\
&= c^*_{uK}(a^*_{x^{-1}H}b_{yH})|_{C_0(\mathcal{A}^0/H)}\, d_{vK}
\end{aligned}$$

Now the product $c^*_{uK}(a^*_{x^{-1}H}b_{yH})|_{C_0(\mathcal{A}^0/H)}\, d_{vK}$ is automatically zero unless $vK = uK$, $xH = yH$ and $vK \subseteq \mathbf{s}(y)H$, in which case we necessarily have that $c^*_{uK}(a^*_{x^{-1}H}b_{yH})|_{C_0(\mathcal{A}^0/H)}\, d_{vK} = (c^*a^*bd)_{vK}$. Hence, we conclude that Ψ_0 preserves the inner products.

Now, if $f_1, f_2 \in C_c(\mathcal{A}/H)$ and $f_3 \in C_c(\mathcal{A}^0/K)$ we have

$$\Psi_0(f_1(f_2 \otimes f_3)) = \Psi_0(f_1f_2 \otimes f_3) = f_1f_2f_3 = f_1\Psi_0(f_2 \otimes f_3)\,.$$

Thus, Ψ_0 preserves the left module actions. Let us now check that Ψ_0 has a dense image in $L^2(\mathcal{A}/K)$. It is enough to prove that all generators $a_{xK} \in C_c(\mathcal{A}/K)$ are in closure of the image of Ψ_0, since their span is dense in $L^2(\mathcal{A}/K)$. To see this, let $\{e^\lambda\}_\lambda$ be an approximate identity of $\mathcal{A}_{\mathbf{s}(x)}$. We have

$$\Psi_0(a_{xH} \otimes e^\lambda_{\mathbf{s}(x)K}) = a_{xH}e^\lambda_{\mathbf{s}(x)K} = (ae^\lambda)_{xK}\,.$$

We then get

$$\begin{aligned}
\|(ae^\lambda)_{xK} - a_{xK}\|^2_{L^2(\mathcal{A}/K)} &= \|(ae^\lambda - a)_{xK}\|^2_{L^2(\mathcal{A}/K)} \\
&= \|\big((ae^\lambda - a)^*(ae^\lambda - a)\big)_{\mathbf{s}(x)K}\|_{C_0(\mathcal{A}^U/K)} \\
&= \|(ae^\lambda - a)^*(ae^\lambda - a)\| \\
&= \|e^\lambda a^*ae^\lambda - e^\lambda a^*a - a^*ae^\lambda + a^*a\|\,.
\end{aligned}$$

Noticing that $a^*a \in \mathcal{A}_{\mathbf{s}(x)}$, we then have that

$$\begin{aligned}
&\leq \|e^\lambda a^*ae^\lambda - e^\lambda a^*a\| + \|-a^*ae^\lambda + a^*a\| \\
&\leq \|a^*ae^\lambda - a^*a\| + \|-a^*ae^\lambda + a^*a\| \\
&\longrightarrow 0\,.
\end{aligned}$$

Thus, we conclude that Ψ_0 has dense range. Hence, from [10, Lemma 2.9], it follows that Ψ_0 extends to an isomorphism of the right-Hilbert $C^*(\mathcal{A}/H)-$

$C_0(\mathcal{A}^0/K)$ bimodules Y and Z. $\square$

Proof of Theorem 8.1.1: The image of $C^*(\mathcal{A}/H)$ in $\mathcal{L}(L^2(\mathcal{A}/K))$ is isomorphic to $C_r^*(\mathcal{A}/H)$ by Lemma 8.1.4. On the other hand, the image of $C^*(\mathcal{A}/H)$ in $\mathcal{L}(L^2(\mathcal{A}/K))$ is simply the completion of $C_c(\mathcal{A}/H)$ as a subalgebra of $C_r^*(\mathcal{A}/K)$. Hence, we conclude that the canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ completes to an embedding of $C_r^*(\mathcal{A}/H)$ into $C_r^*(\mathcal{A}/K)$. $\square$

It follows from Theorem 8.1.1 and Proposition 7.1.3 that for any subgroups $L \subseteq K \subseteq H$ such that $[H : L] < \infty$ the following diagram of canonical embeddings commutes

$$\begin{tikzcd} C_r^*(\mathcal{A}/H) \arrow[r] \arrow[rr, bend right] & C_r^*(\mathcal{A}/K) \arrow[r] & C_r^*(\mathcal{A}/L) \end{tikzcd} .$$

Hence, we have a direct system of C^*-algebras $\{C_r^*(\mathcal{A}/H)\}_{H\in\mathcal{C}}$. Let us denote by $\mathcal{D}_r(\mathcal{A})$ its corresponding C^*-algebraic direct limit

$$\mathcal{D}_r(\mathcal{A}) := \lim_{H\in\mathcal{C}} C_r^*(\mathcal{A}/H)\,. \tag{8.2}$$

We notice that the algebra $\mathcal{D}(\mathcal{A})$ is a dense *-subalgebra of $\mathcal{D}_r(\mathcal{A})$. We now want to show that the action α of G on $\mathcal{D}(\mathcal{A})$ extends to $\mathcal{D}_r(\mathcal{A})$.

Theorem 8.1.5. *The action α of G on $\mathcal{D}(\mathcal{A})$ extends uniquely to an action of G on $\mathcal{D}_r(\mathcal{A})$ and is such that α_g takes $C_r^*(\mathcal{A}/H)$ to $C_r^*(\mathcal{A}/gHg^{-1})$, for any $g \in G$.*

Proof: We have a canonical isomorphism between the right-Hilbert bimodules ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/H)_{C_0(\mathcal{A}^0/H)}$ and ${}_{C^*(\mathcal{A}/gHg^{-1})}L^2(\mathcal{A}/gHg^{-1})_{C_0(\mathcal{A}^0/gHg^{-1})}$, that is determined by the canonical isomorphisms $C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/gHg^{-1})$ and $C_c(\mathcal{A}^0/H) \to C_c(\mathcal{A}^0/gHg^{-1})$ defined by α_g, i.e. defined respectively by

$$a_{xH} \mapsto a_{xg^{-1}gHg^{-1}}\,, \qquad \text{and} \qquad b_{uH} \mapsto b_{ug^{-1}gHg^{-1}}\,,$$

where $x \in X$, $u \in X^0$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_u$. Since $C_r^*(\mathcal{A}/H)$ is the image of $C^*(\mathcal{A}/H)$ inside $\mathcal{L}(L^2(\mathcal{A}/H))$, and similarly for $C_r^*(\mathcal{A}/gHg^{-1})$, we conclude that the isomorphism $C_c(\mathcal{A}/H) \cong C_c(\mathcal{A}/gHg^{-1})$ defined by α_g extends to an isomorphism $C_r^*(\mathcal{A}/H) \cong C_r^*(\mathcal{A}/gHg^{-1})$. Since $C_r^*(\mathcal{A}/gHg^{-1})$ is embedded in $D_r(\mathcal{A})$, we can see α_g as an injective *-homomorphism from $C_r^*(\mathcal{A}/H)$ into $D_r(\mathcal{A})$.

A routine computation shows that the following diagram of canonical injections commutes:

$$\begin{tikzcd} C_r^*(\mathcal{A}/H) \arrow[r] \arrow[dr, "\alpha_g"'] & C_r^*(\mathcal{A}/K) \arrow[d, "\alpha_g"] \\ & D_r(\mathcal{A})\,. \end{tikzcd}$$

Hence, we obtain an injective *-homomorphism from $\mathcal{D}_r(\mathcal{A})$ to itself, which we still denote by α_g, and which extends the usual map α_g from $D(\mathcal{A})$ to itself. It is also clear that this map is surjective, and that for $g, h \in G$ we have $\alpha_{gh} = \alpha_g \circ \alpha_h$, so that we get an action of G on $D_r(\mathcal{A})$ which extends the usual action of G on $D(\mathcal{A})$. □

8.1.2 Maximal completions $C^*(\mathcal{A}/H)$

The purpose of this subsection is to prove the following result:

Theorem 8.1.6. *Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. The canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ completes to a nondegenerate embedding of $C^*(\mathcal{A}/H)$ into $C^*(\mathcal{A}/K)$.*

In order to prove this result we will need to know how to "extend" a representation of $C_c(\mathcal{A}/H)$ to a representation of $C_c(\mathcal{A}/K)$ on a larger Hilbert space.

Definition 8.1.7. Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. Let $\pi : C_c(\mathcal{A}/H) \to B(\mathscr{H})$ be a *-representation. We define the map $\pi^K : C_c(\mathcal{A}/K) \to B(\mathscr{H} \otimes \ell^2(X^0/K))$ by

$$\pi^K(a_{xK})\,(\xi \otimes \delta_{uK}) := \begin{cases} \pi(a_{xH})\xi \otimes \delta_{\mathbf{r}(x)K}\,, & \text{if } uK = \mathbf{s}(x)K \\ 0, & \text{otherwise.} \end{cases} \tag{8.3}$$

Proposition 8.1.8. *The map π^K is a well-defined *-representation.*

Proof: It is clear that the expression that defines $\pi^K(a_{xH})$ defines a linear operator in the inner product space $\mathscr{H} \otimes C_c(X^0/K)$, which is easily observed to be bounded. Thus, $\pi^K(a_{xH}) \in B(\mathscr{H} \otimes \ell^2(X^0/K))$.

It is clear that expression (8.3) defines a linear mapping π^K on $C_c(\mathcal{A}/K)$, so that we only need to see that it preserves products and the involution. To see that it preserves products, consider two elements of the form a_{xK} and b_{yK}. There are two cases to consider: either $\mathbf{r}(y) \in \mathbf{s}(x)K$ or $\mathbf{r}(y) \notin \mathbf{s}(x)K$.

In the second case, we have $a_{xK}b_{yK} = 0$ and thus $\pi^K(a_{xK}b_{yK}) = 0$. But also $\pi^K(a_{xK})\pi^K(b_{yK}) = 0$, because for any vector $\xi \otimes \delta_{uK}$ we have that $\pi^K(b_{yK})(\xi \otimes \delta_{uK})$ is either zero or equal to $\pi(b_{yK})\xi \otimes \delta_{\mathbf{r}(y)K}$, and therefore we always have $\pi^K(a_{xK})\pi^K(b_{yK})(\xi \otimes \delta_{uK}) = 0$.

In the first case we have

$$
\begin{aligned}
\pi^K(a_{xK}b_{yK})\,(\xi \otimes \delta_{uK}) &= \pi^K((ab)_{x\widetilde{k}yK})\,(\xi \otimes \delta_{uK}) \\
&= \begin{cases} \pi((ab)_{x\widetilde{k}yH})\xi \otimes \delta_{\mathbf{r}(x\widetilde{k}y)K}\,, & \text{if } \ uK = \mathbf{s}(\widetilde{xky})K \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \pi(a_{xH})\pi(b_{yH})\xi \otimes \delta_{\mathbf{r}(x)K}\,, & \text{if } \ uK = \mathbf{s}(y)K \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \pi^K(a_{xK})\,(\pi(b_{yH})\xi \otimes \delta_{\mathbf{s}(x)K})\,, & \text{if } \ uK = \mathbf{s}(y)K \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \pi^K(a_{xK})\,(\pi(b_{yH})\xi \otimes \delta_{\mathbf{r}(y)K})\,, & \text{if } \ uK = \mathbf{s}(y)K \\ 0, & \text{otherwise.} \end{cases} \\
&= \pi^K(a_{xK})\pi^K(b_{yK})\,(\xi \otimes \delta_{uK})\,.
\end{aligned}
$$

In both cases we have $\pi^K(a_{xK}b_{yK}) = \pi^K(a_{xK})\pi^K(b_{yK})$, hence π^K preserves products. Let us now check that it preserves the involution. We have

$$
\begin{aligned}
&\langle \pi^K(a_{xK})\,(\xi\otimes\delta_{uK})\,,\,\eta\otimes\delta_{vK}\rangle \;= \\
= \;& \begin{cases} \langle \pi(a_{xH})\xi\otimes\delta_{\mathbf{r}(x)K}\,,\,\eta\otimes\delta_{vK}\rangle\,, & \text{if } uK=\mathbf{s}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \;& \begin{cases} \langle \pi(a_{xH})\xi\,,\,\eta\rangle\,, & \text{if } uK=\mathbf{s}(x)K \text{ and } vK=\mathbf{r}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \;& \begin{cases} \langle \xi\,,\,\pi(a^*_{x^{-1}H})\eta\rangle\,, & \text{if } uK=\mathbf{s}(x)K \text{ and } vK=\mathbf{r}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \;& \begin{cases} \langle \xi\otimes\delta_{uK}\,,\,\pi(a^*_{x^{-1}H})\eta\otimes\delta_{\mathbf{s}(x)K}\rangle\,, & \text{if } vK=\mathbf{r}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \;& \langle \xi\otimes\delta_{uK}\,,\,\pi^K(a^*_{x^{-1}K})\,(\eta\otimes\delta_{vK})\rangle\,.
\end{aligned}
$$

Hence, we conclude that $\pi^K(a_{xK})^* = \pi^K((a_{xK})^*)$, and therefore π^K preserves the involution. Hence, π^K is a *-representation. $\square$

Lemma 8.1.9. *Let us denote by $\delta_{uH}\in\ell^2(X^0/K)$ the vector*

$$\delta_{uH} := \sum_{[h]\in\mathcal{S}_u\backslash H/K} \delta_{uhK}\,. \tag{8.4}$$

The map π^K satisfies

$$\pi^K(a_{xH})\,(\xi\otimes\delta_{uH}) := \begin{cases} \pi(a_{xH})\xi\otimes\delta_{\mathbf{r}(x)H}\,, & \textit{if } uH=\mathbf{s}(x)H, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: We have

$$\pi^K(a_{xH})\,(\xi\otimes\delta_{uH}) \;=\; \sum_{[h]\in\mathcal{S}_x\backslash H/K}\;\sum_{[h']\in\mathcal{S}_u\backslash H/K} \pi^K(a_{xhK})\,(\xi\otimes\delta_{uh'K})\,,$$

from which we see that, if $uH\neq\mathbf{s}(x)H$ then $\pi^K(a_{xH})\,(\xi\otimes\delta_{uH})=0$. On the other hand, if $uH = \mathbf{s}(x)H$, then we have

$$\begin{aligned}
\pi^K(a_{xH})\,(\xi \otimes \delta_{\mathbf{s}(x)H}) &= \sum_{[h]\in\mathcal{S}_x\backslash H/K}\ \sum_{[h']\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi^K(a_{xhK})\,(\xi \otimes \delta_{\mathbf{s}(x)h'K}) \\
&= \sum_{[h]\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K}\ \sum_{[h']\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi^K(a_{xhK})\,(\xi \otimes \delta_{\mathbf{s}(x)h'K}) \\
&= \sum_{[h]\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi(a_{xH})\xi \otimes \delta_{\mathbf{r}(x)hK} \\
&= \sum_{[h]\in\mathcal{S}_{\mathbf{r}(x)}\backslash H/K} \pi(a_{xH})\xi \otimes \delta_{\mathbf{r}(x)hK} \\
&= \pi(a_{xH})\xi \otimes \delta_{\mathbf{r}(x)H}\,.
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 8.1.6: In order to prove this statement we have to show that for any $f \in C_c(\mathcal{A}/H)$ we have $\|f\|_{C^*(\mathcal{A}/K)} = \|f\|_{C^*(\mathcal{A}/H)}$. Since we are viewing $C_c(\mathcal{A}/H)$ as a *-subalgebra of $C_c(\mathcal{A}/K)$ we automatically have the inequality

$$\|f\|_{C^*(\mathcal{A}/K)} \leq \|f\|_{C^*(\mathcal{A}/H)}\,.$$

In order to prove the converse inequality, it suffices to prove that

$$\|\pi(f)\| \leq \|\pi^K(f)\|\,, \tag{8.5}$$

for any nondegenerate *-representation π of $C_c(\mathcal{A}/H)$, because, since π is arbitrary, this clearly implies that $\|f\|_{C^*(\mathcal{A}/H)} \leq \|f\|_{C^*(\mathcal{A}/K)}$. Let us then prove inequality (8.5).

We can write any element $f \in C_c(\mathcal{A}/H)$ as $f = \sum_{xH\in X/H}(f(xH))_{xH}$. Furthermore we can split this sum according to the ranges of elements, i.e.

$$f = \sum_{xH\in X/H} (f(xH))_{xH} \;=\; \sum_{vH\in X^0/H}\ \sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} (f(xH))_{xH}\,.$$

Suppose $\pi : C_c(\mathcal{A}/H) \to B(\mathscr{H})$ is a *-representation and $\xi \in \mathscr{H}$ is a vector of norm one. We have

$$\|\pi\Big(\sum_{xH\in X/H} (f(xH))_{xH}\Big)\xi\|^2 \;=\; \|\sum_{vH\in X^0/H} \pi\Big(\sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} (f(xH))_{xH}\Big)\xi\|^2\,.$$

For different units $vH \in X^0/H$, the elements $\pi\Big(\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} (f(xH))_{xH}\Big)\xi$ are easily seen to be orthogonal, so that

$$= \sum_{vH\in X^0/H} \|\pi\Big(\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} (f(xH))_{xH}\Big)\xi\|^2$$

$$= \sum_{vH\in X^0/H} \|\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} \pi((f(xH))_{xH})\xi\|^2$$

In the notation of (8.4), let $\delta_{uH} := \sum_{[h]\in\mathcal{S}_u\backslash H/K} \delta_{uhK}$. Let us denote by C_u the number of elements of $\mathcal{S}_u\backslash H/K$. We have

$$= \sum_{vH\in X^0/H} \frac{1}{C_v}\|\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} \pi((f(xH))_{xH})\xi\otimes\delta_{vH}\|^2$$

$$= \|\sum_{vH\in X^0/H}\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} \frac{1}{C_v}\pi((f(xH))_{xH})\xi\otimes\delta_{vH}\|^2$$

$$= \|\sum_{vH\in X^0/H}\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} \frac{1}{C_v}\,\pi((f(xH))_{xH})\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{vH}\|^2 .$$

By Lemma 8.1.9 we have that

$$= \|\sum_{vH\in X^0/H}\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} \frac{1}{C_v}\,\pi^K((f(xH))_{xH})\left(\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2 ,$$

and since $\mathcal{S}_{\mathbf{s}(x)}\backslash H/K = \mathcal{S}_{\mathbf{r}(x)}\backslash H/K$, we get that $C_{\mathbf{s}(x)} = C_{\mathbf{r}(x)}$. Thus,

$$= \|\sum_{vH\in X^0/H}\sum_{\substack{xH\in X/H\\ \mathbf{r}(x)H=vH}} \frac{1}{C_{\mathbf{s}(x)}}\,\pi^K((f(xH))_{xH})\left(\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2$$

$$= \|\sum_{xH\in X/H} \frac{1}{C_{\mathbf{s}(x)}}\,\pi^K((f(xH))_{xH})\left(\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2$$

$$= \|\sum_{xH\in X/H} \pi^K((f(xH))_{xH})\left(\frac{1}{C_{s(x)}}\,\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2 .$$

Similarly as we did for ranges, we can split the sum $\sum_{xH\in X/H}(f(xH))_{xH}$ according to sources. In this way, since this sum is finite, there is a finite number of units $u_1H, \ldots, u_nH \in X^0/H$, which we assume to be pairwise different, such that we can write

$$\sum_{xH \in X/H} (f(xH))_{xH} = \sum_{i=1}^{n} \sum_{\substack{xH \in X/H \\ \mathbf{s}(x)H = u_iH}} (f(xH))_{xH} \,.$$

By Lemma 8.1.9 we see that $\pi^K((f(xH))_{xH})\Big(\widetilde{\pi}(1_{u_iH})\xi \otimes \delta_{u_iH}\Big) = 0$ unless $\mathbf{s}(x)H = u_iH$. Hence we get

$$\begin{aligned}
& \| \sum_{xH \in X/H} \pi^K((f(xH))_{xH}) \Big(\frac{1}{C_{s(x)}} \widetilde{\pi}(1_{\mathbf{s}(x)H})\xi \otimes \delta_{\mathbf{s}(x)H}\Big) \|^2 \\
= \; & \| \sum_{xH \in X/H} \pi^K((f(xH))_{xH}) \big(\sum_{i=1}^{n} \frac{1}{C_{u_i}} \widetilde{\pi}(1_{u_iH})\xi \otimes \delta_{u_iH}\big) \|^2 \\
= \; & \|\pi^K\Big(\sum_{xH \in X/H} (f(xH))_{xH}\Big) \big(\sum_{i=1}^{n} \frac{1}{C_{u_i}} \widetilde{\pi}(1_{u_iH})\xi \otimes \delta_{u_iH}\big) \|^2 \,.
\end{aligned}$$

We now notice that, since we are assuming ξ to be of norm one, it follows that the vector

$$\sum_{i=1}^{n} \frac{1}{C_{u_i}} \widetilde{\pi}(1_{u_iH})\xi \otimes \delta_{u_iH} \,,$$

also has norm less or equal to one, because

$$\begin{aligned}
\| \sum_{i=1}^{n} \frac{1}{C_{u_i}} \widetilde{\pi}(1_{u_iH})\xi \otimes \delta_{u_iH}\|^2 &= \sum_{i=1}^{n} \| \frac{1}{C_{u_i}} \widetilde{\pi}(1_{u_iH})\xi \otimes \delta_{u_iH}\|^2 \\
&= \sum_{i=1}^{n} \|\widetilde{\pi}(1_{u_iH})\xi\|^2 \\
&= \|\widetilde{\pi}\Big(\sum_{i=1}^{n} 1_{u_iH}\Big)\xi\|^2 \\
&\leq \|\xi\|^2 \\
&= 1 \,.
\end{aligned}$$

Hence, taking the supremum over vectors ξ of norm one, we immediately get the inequality

$$\|\pi(f)\| \;\leq\; \|\pi^K(f)\| \,.$$

As we explained earlier, this proves that we get an embedding of $C^*(\mathcal{A}/H)$ into $C^*(\mathcal{A}/K)$. $\square$

It follows from 8.1.6 that $\{C^*(\mathcal{A}/H)\}_{H\in\mathcal{C}}$ is a direct system of C^*-algebras. Let us denote by $\mathcal{D}_{\max}(\mathcal{A})$ its corresponding C^*-algebraic direct limit

$$\mathcal{D}_{\max}(\mathcal{A}) := \lim_{\substack{\longrightarrow \\ H\in\mathcal{C}}} C^*(\mathcal{A}/H)\,, \tag{8.6}$$

We notice that the algebra $\mathcal{D}(\mathcal{A})$ is a dense *-subalgebra of $\mathcal{D}_{\max}(\mathcal{A})$. We now want to show that the action α of G on $\mathcal{D}(\mathcal{A})$ extends to $\mathcal{D}_{\max}(\mathcal{A})$.

Theorem 8.1.10. *The action α of G on $\mathcal{D}(\mathcal{A})$ extends uniquely to an action of G on $\mathcal{D}_{\max}(\mathcal{A})$ and is such that α_g takes $C^*(\mathcal{A}/H)$ to $C^*(\mathcal{A}/gHg^{-1})$, for any $g \in G$.*

Proof: Since α_g is a *-isomorphism between $C_c(\mathcal{A}/H)$ and $C_c(\mathcal{A}/gHg^{-1})$, it necessarily extends to a *-isomorphism between the enveloping C^*-algebras $C^*(\mathcal{A}/H)$ and $C^*(\mathcal{A}/gHg^{-1})$. Since $C^*(\mathcal{A}/gHg^{-1})$ is embedded in $D_{\max}(\mathcal{A})$, we can see α_g as an injective *-homomorphism from $C^*(\mathcal{A}/H)$ into $D_{\max}(\mathcal{A})$.

A routine computation shows that the following diagram of canonical injections commutes:

$$\begin{tikzcd}
C^*(\mathcal{A}/H) \arrow[r] \arrow[dr, "\alpha_g"'] & C^*(\mathcal{A}/K) \arrow[d, "\alpha_g"] \\
& D_{\max}(\mathcal{A})\,.
\end{tikzcd}$$

Hence, we obtain an injective *-homomorphism from $\mathcal{D}_{\max}(\mathcal{A})$ to itself, which we still denote by α_g, and which extends the usual map α_g from $D(\mathcal{A})$ to itself. It is also clear that this map is surjective, and that for $g, h \in G$ we have $\alpha_{gh} = \alpha_g \circ \alpha_h$, so that we get an action of G on $D_{\max}(\mathcal{A})$ which extends the usual action of G on $D(\mathcal{A})$. $\square$

8.2 Reduced C^*-crossed products

We now want to define reduced C^*-norms in the *-algebraic crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. Since $C_c(\mathcal{A}/\Gamma)$ admits several canonical C^*-completions one should expect that there are several reduced C^*-norms we can give to $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, which give rise to different reduced C^*-crossed products, as for example $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. We will treat in this section all these different reduced C^*-norms (and reduced C^*-crossed products) in a single approach, and for that the notion we need is that of a *α-extendable* C^*-norm on $\mathcal{D}(\mathcal{A})$:

Definition 8.2.1. A C^*-norm $\|\cdot\|_\tau$ in $\mathcal{D}(\mathcal{A})$ is said to be *α-extendable* if the action α of G on $\mathcal{D}(\mathcal{A})$ extends to $\mathcal{D}_\tau(\mathcal{A})$, the completion of $\mathcal{D}(\mathcal{A})$ with respect to the norm $\|\cdot\|_\tau$. In other words, if for every $g \in G$ the automorphism α_g of $\mathcal{D}(\mathcal{A})$ is continuous with respect to $\|\cdot\|_\tau$.

Definition 8.2.2. Let $\|\cdot\|_\tau$ be an α-extendable C^*-norm in $\mathcal{D}(\mathcal{A})$ and let us denote by $\mathcal{D}_\tau(\mathcal{A})$ and $C^*_\tau(\mathcal{A}/\Gamma)$ the completions of $\mathcal{D}(\mathcal{A})$ and $C_c(\mathcal{A}/\Gamma)$, respectively, with respect to the norm $\|\cdot\|_\tau$. We define the norm $\|\cdot\|_{\tau,r}$ in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ by

$$\|f\|_{\tau,r} := \sup_{\pi \in R(\mathcal{D}_\tau(\mathcal{A}))} \|[\pi_\alpha \times (1 \otimes \rho)](f)\| ,$$

where the supremum is taken over the class $R(\mathcal{D}_\tau(\mathcal{A}))$ of all nondegenerate *-representations of $\mathcal{D}_\tau(\mathcal{A})$. The completion of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ with respect to this norm shall be denoted by $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and referred to as the *reduced crossed product* of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ).

Before we prove that $\|\cdot\|_\tau$ is indeed a C^*-norm, let us first look at the two main instances we have in mind, which arise when $C^*_\tau(\mathcal{A}/\Gamma)$ is $C^*_r(\mathcal{A}/\Gamma)$ or $C^*(\mathcal{A}/\Gamma)$. It is not obvious from the start that there exists a C^*-norm $\|\cdot\|_\tau$ in $\mathcal{D}(\mathcal{A})$ whose restriction to $C_c(\mathcal{A}/\Gamma)$ will give the reduced or the maximal C^*-norm in $C_c(\mathcal{A}/\Gamma)$, but this is indeed the case from what we proved in the preliminary sections 8.1.1 and 8.1.2:

- *For $C^*_r(\mathcal{A}/\Gamma)$:*

  As described in Section 8.1.1, we can form the C^*-algebraic direct limit $\mathcal{D}_r(\mathcal{A}) = \lim_{H \in \mathcal{C}} C^*_r(\mathcal{A}/H)$, which contains $\mathcal{D}(\mathcal{A})$ as a dense *-subalgebra. Taking $\|\cdot\|_\tau$ to be the C^*-norm $\|\cdot\|_r$ of $\mathcal{D}_r(\mathcal{A})$, we see that $C^*_\tau(\mathcal{A}/\Gamma) = C^*_r(\mathcal{A}/\Gamma)$. The norm $\|\cdot\|_r$ is α-extendable because of Theorem 8.1.5.

- *For $C^*(\mathcal{A}/\Gamma)$:*

  As described in Section 8.1.2, we can form the C^*-algebraic direct limit $\mathcal{D}_{\max}(\mathcal{A}) = \lim_{H \in \mathcal{C}} C^*(\mathcal{A}/H)$, which contains $\mathcal{D}(\mathcal{A})$ as a dense

*-subalgebra. Taking $\|\cdot\|_\tau$ to be the C^*-norm $\|\cdot\|_{\max}$ of $\mathcal{D}_{\max}(\mathcal{A})$, we see that $C^*_\tau(\mathcal{A}/\Gamma) = C^*(\mathcal{A}/\Gamma)$. The norm $\|\cdot\|_{\max}$ is α-extendable because of Theorem 8.1.10.

Lemma 8.2.3. *If $\pi : \mathcal{D}(\mathcal{A}) \to B(\mathscr{H})$ is a nondegenerate *-representation which is continuous with respect to an α-extendable norm $\|\cdot\|_\tau$ in $\mathcal{D}(\mathcal{A})$, then π_α is a representation of $C_c(\mathcal{A}/\Gamma)$ which is continuous with respect to the norm $\|\cdot\|_\tau$ as well.*

Proof: Let $f \in C_c(\mathcal{A}/\Gamma)$. We have

$$\begin{aligned}
\|\pi_\alpha(f)\big(\sum_{[h]\in G/\Gamma} \xi_{h\Gamma}\otimes\delta_{h\Gamma}\big)\|^2 &= \|\sum_{[h]\in G/\Gamma} \pi(\alpha_h(f))\xi_{h\Gamma}\otimes\delta_{h\Gamma}\|^2 \\
&= \sum_{[h]\in G/\Gamma} \|\pi(\alpha_h(f))\xi_{h\Gamma}\|^2 \\
&\leq \sum_{[h]\in G/\Gamma} \|\pi(\alpha_h(f))\|^2\|\xi_{h\Gamma}\|^2 \\
&\leq \sum_{[h]\in G/\Gamma} \|\alpha_h(f)\|_\tau^2\,\|\xi_{h\Gamma}\|^2
\end{aligned}$$

Since the action is α-extendable we have that

$$\|\alpha_h(f)\|_\tau = \|f\|_\tau\,.$$

Hence we have

$$\begin{aligned}
\|\pi_\alpha(f)\big(\sum_{[h]\in G/\Gamma} \xi_{h\Gamma}\otimes\delta_{h\Gamma}\big)\|^2 &\leq \sum_{[h]\in G/\Gamma} \|f\|_\tau^2\|\xi_{h\Gamma}\|^2 \\
&= \|f\|_\tau^2\,\|\sum_{[h]\in G/\Gamma} \xi_{h\Gamma}\otimes\delta_{h\Gamma}\|^2\,.
\end{aligned}$$

Hence, π_α is continuous with respect to the norm $\|\cdot\|_\tau$. □

Proposition 8.2.4. *$\|\cdot\|_{\tau,r}$ is a well-defined C^*-norm on $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$.*

Proof: First we must show that the supremum in the definition of $\|\cdot\|_{\tau,r}$ is bounded. Given a *-representation π of $\mathcal{D}_\tau$ we have, by Lemma 8.2.3, that

$$\begin{aligned} & \|[\pi_\alpha \times (1 \otimes \rho)](f)\| \\ \leq & \sum_{[g] \in \Gamma \backslash G/\Gamma} \sum_{x\Gamma^g \in X/\Gamma^g} \|\pi_\alpha\big((f(g\Gamma)(x\Gamma^g))_{x\Gamma}\big)\| \|(1 \otimes \rho)\,(\Gamma g \Gamma)\| \\ \leq & \sum_{[g] \in \Gamma \backslash G/\Gamma} \sum_{x\Gamma^g \in X/\Gamma^g} \|(f(g\Gamma)(x\Gamma^g))_{x\Gamma}\|_\tau \, \|\Gamma g \Gamma\|_{C^*_\tau(G,\Gamma)} \,. \end{aligned}$$

Since the right hand side is finite and does not depend on π, we conclude that $\|f\|_{\tau,r}$ is bounded by this number.

It is clear from the definition and the above paragraph that $\|\cdot\|_{\tau,r}$ is C^*-seminorm. To prove that it is actually a C^*-norm it is enough to prove that if π is a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$, then $\pi_\alpha \times (1 \otimes \rho)$ is a faithful *-representation of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. Let us then prove this claim. Suppose $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is such that $[\pi_\alpha \times (1 \otimes \rho)](f) = 0$. Then, for every $\xi \otimes \delta_{h\Gamma} \in \mathscr{H} \otimes \ell^2(G/\Gamma)$ we have

$$0 \;=\; [\pi_\alpha \times (1 \otimes \rho)](f)(\xi \otimes \delta_{h\Gamma}) \;=\; \sum_{[g] \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\alpha_g(f(g^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \,.$$

In particular, for $g\Gamma = \Gamma$, we have $\pi(f(h\Gamma))\xi = 0$, and since this holds for every $\xi \in \mathscr{H}$ we have $\pi(f(h\Gamma)) = 0$. Now, since π is a faithful *-representation, it follows that $f(h\Gamma) = 0$. Since this holds for every $h\Gamma \in G/\Gamma$, we have $f = 0$, i.e. $\pi_\alpha \times (1 \otimes \rho)$ is injective. $\square$

The next result explains why we call the completion of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ under the norm $\|\cdot\|_{\tau,r}$ the reduced crossed product of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G,Γ) and justifies also the notation $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ chosen to denote this completion.

Proposition 8.2.5. *The restriction of the norm $\|\cdot\|_{\tau,r}$ of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ to $C_c(\mathcal{A}/\Gamma)$ is precisely the norm $\|\cdot\|_\tau$ of $C_c(\mathcal{A}/\Gamma)$. Hence, the embedding $C_c(\mathcal{A}/\Gamma) \to C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ completes to an embedding $C^*_\tau(\mathcal{A}/\Gamma) \to C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.*

Proof: Let $\pi : \mathcal{D}_\tau(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. From Lemma 8.2.3 we have

$$\|[\pi_\alpha \times (1 \otimes \rho)](f)\| = \|\pi_\alpha(f)\| \leq \|f\|_\tau \,,$$

for every $f \in C_c(\mathcal{A}/\Gamma)$, and therefore

$$\|f\|_{\tau,r} \leq \|f\|_\tau \,.$$

We now wish to prove the converse inequality. Let $\pi : \mathcal{D}_\tau(\mathcal{A}) \to B(\mathscr{H})$ be a faithful nondegenerate *-representation. For any $f \in C_c(\mathcal{A}/\Gamma)$ we have

$$\begin{aligned}
\|f\|_\tau &= \|\pi(f)\| = \sup_{\|\xi\|=1} \|\pi(f)\xi\| \\
&= \sup_{\|\xi\|=1} \|\pi_\alpha(f)(\xi \otimes \delta_\Gamma)\| \\
&\leq \sup_{\|\zeta\|=1} \|\pi_\alpha(f)\zeta\| = \|\pi_\alpha(f)\| \\
&= \|[\pi_\alpha \times (1 \otimes \rho)](f)\| \leq \|f\|_{\tau,r} \,,
\end{aligned}$$

thus proving the converse inequality. We conclude that

$$\|f\|_{\tau,r} = \|f\|_\tau \,,$$

for any $f \in C_c(\mathcal{A}/\Gamma)$ and this finishes the proof. □

An important feature of reduced crossed products by groups $A \times_r G$ is the existence of faithful conditional expectation onto A. We will now explain how this holds as well for reduced crossed products by Hecke pairs, with somewhat analogous proofs. The goal is to prove Theorem 8.2.7 bellow, and for that we follow closely the approach presented in [36] in the case of groups.

Proposition 8.2.6. *For every $g\Gamma \in G/\Gamma$ the map $E_{g\Gamma}$ defined by*

$$\begin{aligned}
E_{g\Gamma} : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma &\longrightarrow C^*_\tau(\mathcal{A}/\Gamma^g) \\
E_{g\Gamma}(f) &:= f(g\Gamma) \,.
\end{aligned}$$

is linear and continuous with respect to the norm $\|\cdot\|_{\tau,r}$.

Before we give a proof of the result above we need to set some notation. For each element $g\Gamma \in G/\Gamma$ we will denote by $\sigma_{g\Gamma}$ the Hilbert space isometry $\sigma_{g\Gamma} : \mathscr{H} \to \mathscr{H} \otimes \ell^2(G/\Gamma)$ defined by

$$\sigma_{g\Gamma}(\xi) := \xi \otimes \delta_{g\Gamma} \,. \tag{8.7}$$

Proof of Proposition 8.2.6: Let π be a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$. It is easily seen that $\sigma_\Gamma^* [\pi \times (1 \otimes \rho)](f) \, \sigma_{g\Gamma} = \Delta(g)^{\frac{1}{2}} \, \pi(f(g\Gamma))$. Hence we have

$$\begin{aligned} \|E_{g\Gamma}(f)\|_\tau &= \|f(g\Gamma)\|_\tau \;\; = \;\; \|\pi(f(g\Gamma))\| \\ &= \|\Delta(g^{-1})^{\frac{1}{2}} \, \sigma_\Gamma^* \, [\pi_\alpha \times (1 \otimes \rho)](f) \, \sigma_{g\Gamma}\| \\ &\leq \Delta(g^{-1})^{\frac{1}{2}} \, \|[\pi_\alpha \times (1 \otimes \rho)](f)\| \\ &\leq \Delta(g^{-1})^{\frac{1}{2}} \, \|f\|_{\tau,r} \, . \end{aligned}$$

This finishes the proof. □

We shall henceforward make no distinction of notation between the maps $E_{g\Gamma}$ defined on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and their extension to $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

The following result is of particular importance in theory of reduced C^*-crossed products. Analogously to the case of groups, it reveals two important features of reduced C^*-crossed products by Hecke pairs: the fact that every element of a reduced crossed product is uniquely described in terms of its coefficients (determined by the $E_{g\Gamma}$); and the fact that E_Γ is a faithful conditional expectation.

Theorem 8.2.7. *We have*

i) If $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $E_{g\Gamma}(f) = 0$ for all $g\Gamma \in G/\Gamma$, then $f = 0$.

ii) E_Γ is a faithful conditional expectation of $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ onto $C_\tau^*(\mathcal{A}/\Gamma)$.

We start with the following auxiliary result:

Lemma 8.2.8. *Let π be a nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$. For all $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ we have*

$$\sigma_{g\Gamma}^* \, [\pi \times (1 \otimes \rho)](f) \, \sigma_{h\Gamma} = \Delta(g^{-1}h)^{\frac{1}{2}} \, \pi(\alpha_g(E_{g^{-1}h\Gamma}(f))) \, . \qquad (8.8)$$

Proof: We notice that equality (8.8) above holds for any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, following the definitions of the maps $E_{t\Gamma}$, $[\pi_\alpha \times (1 \otimes \rho)](f)$ and $\sigma_{t\Gamma}$, with $t\Gamma \in G/\Gamma$. By continuity, it follows readily that the equality must hold for every $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. $\square$

Proof of Theorem 8.2.7: *i)* Let $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Suppose $E_{g\Gamma}(f) = 0$ for all $g\Gamma \in G/\Gamma$. Then, for any given nondegenerate *-representation π of $\mathcal{D}_\tau(\mathcal{A})$ we have, by Lemma 8.2.8, that $\sigma^*_{g\Gamma}\, [\pi_\alpha \times (1 \otimes \rho)](f)\, \sigma_{h\Gamma} = 0$ for all $g\Gamma, h\Gamma \in G/\Gamma$. Hence, $[\pi_\alpha \times (1 \otimes \rho)](f) = 0$. Since, this is true for any π, we must have $\|f\|_{\tau,r} = 0$, i.e. $f = 0$.

ii) Let us first prove that E_Γ is a conditional expectation, i.e. E_Γ is an idempotent, positive, $C^*_\tau(\mathcal{A}/\Gamma)$-linear map.

If $f \in C_c(\mathcal{A}/\Gamma)$ then it is clear that $E_\Gamma(f) = f$. By continuity and Proposition 8.2.5 it follows that $E_\Gamma(f) = f$ for all $f \in C^*_\tau(\mathcal{A}/\Gamma)$. Thus, E_Γ is idempotent.

Suppose now that $f \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$. We have

$$\begin{aligned} E_\Gamma(f^* * f) &= (f^* * f)(\Gamma) = \sum_{[h] \in G/\Gamma} f^*(h\Gamma)\alpha_h(f(h^{-1}\Gamma)) \\ &= \sum_{[h] \in G/\Gamma} \Delta(h^{-1})\, \alpha_h(f(h^{-1}\Gamma))^* \alpha_h(f(h^{-1}\Gamma)) \geq 0 \end{aligned}$$

By continuity it follows that $E_\Gamma(f^* * f) \geq 0$ for all $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, i.e. E_Γ is positive. It remains to show that E_Γ is $C^*_\tau(\mathcal{A}/\Gamma)$-linear. We recall that we see $C_c(\mathcal{A}/\Gamma)$ as a *-subalgebra of $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ in the following way: an element $f \in C_c(\mathcal{A}/\Gamma)$ is identified with the element $F \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ with support in Γ and such that $F(\Gamma) = f$. For any $f \in C_c(\mathcal{A}/\Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ we have

$$\begin{aligned} E_\Gamma(f * f_2) &= (F * f_2)(\Gamma) = \sum_{[h] \in G/\Gamma} F(h\Gamma)\alpha_h(f_2(h^{-1}\Gamma)) \\ &= F(\Gamma) f_2(\Gamma) = f\, E_\Gamma(f_2)\,, \end{aligned}$$

and similarly we get $E_\Gamma(f_2 * f) = E_\Gamma(f_2)\, f$. Once again by continuity we conclude that the same equalities hold for $f \in C^*_\tau(\mathcal{A}/\Gamma)$ and $f_2 \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Thus, E_Γ is a conditional expectation.

Let us now prove that E_Γ is faithful. For any $f \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ we have (where the first equality was computed above)

$$\begin{aligned} E_\Gamma(f^* * f) &= \sum_{[h] \in G/\Gamma} \Delta(h^{-1})\, \alpha_h(f(h^{-1}\Gamma))^* \alpha_h(f(h^{-1}\Gamma)) \\ &= \sum_{[h] \in G/\Gamma} \Delta(h^{-1})\, \alpha_h(E_{h-1\Gamma}(f))^* \alpha_h(E_{h-1\Gamma}(f))\,. \end{aligned}$$

Hence, we have $E_\Gamma(f^* * f) \geq \Delta(h^{-1})\, \alpha_h(E_{h-1\Gamma}(f))^* \alpha_h(E_{h-1\Gamma}(f))$ for each $h\Gamma \in G/\Gamma$,. By continuity this inequality holds for every $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, and therefore if $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ is such that $E_\Gamma(f^* * f) = 0$, then $E_{g\Gamma}(f) = 0$ for all $g\Gamma \in G/\Gamma$. Hence, by part *i*), we conclude that $f = 0$. Thus, E_Γ is faithful. $\square$

The next result shows, like in crossed products by groups, that to define the norm $\|\cdot\|_{\tau,r}$ of the reduced crossed product $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ we only need to start with a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$, instead of taking the supremum over all nondegenerate *-representations of $\mathcal{D}_\tau(\mathcal{A})$.

Theorem 8.2.9. *Let $\pi : \mathcal{D}_\tau(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. We have that*

i) If $\pi_\alpha : C^*_\tau(\mathcal{A}/\Gamma) \to B(\mathscr{H} \otimes \ell^2(G/\Gamma))$ is faithful, then $[\pi_\alpha \times (1 \otimes \rho)]$ is a faithful *-representation of $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Consequently,

$$\|f\|_{\tau,r} = \|[\pi_\alpha \times (1 \otimes \rho)](f)\|\,,$$

for all $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

ii) If π is faithful, then π_α is faithful.

Proof: Let us prove *i*) first. Suppose π_α is faithful as a *-representation of $C^*_\tau(\mathcal{A}/\Gamma)$. Let $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ be such that $[\pi \times (1 \otimes \rho)](f) = 0$. Then, of course, $[\pi_\alpha \times (1 \otimes \rho)](f^* * f) = 0$ and we have

$$\begin{aligned} 0 &= \sigma^*_{g\Gamma}\, [\pi_\alpha \times (1 \otimes \rho)](f^* * f)\, \sigma_{g\Gamma} \;=\; \pi(\alpha_g(E_\Gamma(f^* * f))) \\ &= \sigma^*_{g\Gamma}\, \pi_\alpha(E_\Gamma(f^* * f))\, \sigma_{g\Gamma}\,. \end{aligned}$$

This implies that $\pi_\alpha(E_\Gamma(f^* * f)) = 0$, i.e. $E_\Gamma(f^* * f) = 0$, and since E_Γ is a faithful conditional expectation we have $f^* * f = 0$, i.e. $f = 0$. Thus, $\pi_\alpha \times (1 \otimes \rho)$ is faithful.

Let us now prove claim *ii*). We know that π_α, as a *-representation of $C_c(\mathcal{A}/\Gamma)$, is given by

$$\pi_\alpha(f)\,(\xi \otimes \delta_{g\Gamma}) = \pi(\alpha_g(f))\xi \otimes \delta_{g\Gamma}\,,$$

By continuity the same expression holds for $f \in C^*_\tau(\mathcal{A}/\Gamma)$. Now suppose that $\pi_\alpha(f) = 0$ for some $f \in C^*_\tau(\mathcal{A}/\Gamma)$. Then, by the above expression, we have

$\pi(f) = 0$. Since π is faithful we must have $f = 0$. Thus, π_α is faithful. $\square$

Another feature of reduced C^*-crossed products by groups $A \times_r G$ is the fact that the reduced C^*-algebra of the group is always canonically embedded in the multiplier algebra $M(A \times_r G)$. The same is true in the Hecke pair case as we now show:

Proposition 8.2.10. *There is a unique embedding of the reduced Hecke C^*-algebra $C_r^*(G,\Gamma)$ into $M(C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)$ extending the action of $\mathcal{H}(G,\Gamma)$ on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.*

Proof: Let us first see that the action of $\mathcal{H}(G,\Gamma)$ on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is continuous with respect to the norm $\|\cdot\|_{\tau,r}$, so that it extends uniquely to an action of $\mathcal{H}(G,\Gamma)$ on $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

Let π be a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$. From Theorem 8.2.9 we know that $\pi_\alpha \times (1 \otimes \rho)$ is also faithful. For $f_1 \in \mathcal{H}(G,\Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, we have

$$\begin{aligned} \|f_1 * f_2\|_{\tau,r} &= \|[\pi_\alpha \times (1 \otimes \rho)](f_1 * f_2)\| \\ &\leq \|(1 \otimes \rho)(f_1)\| \|[\pi_\alpha \times (1 \otimes \rho)](f_2)\| \\ &= \|\rho(f_1)\| \|f_2\|_{\tau,r} \,. \end{aligned}$$

Thus, the action of $\mathcal{H}(G,\Gamma)$ on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ extends uniquely to an action on $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, or in other words, we have an embedding of $\mathcal{H}(G,\Gamma)$ into $M(C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)$. We now want to prove that this embedding extends to an embedding of $C_r^*(G,\Gamma)$ into the same multiplier algebra. For that it is enough to prove that

$$\|f\|_{M(C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)} = \|f\|_{C_r^*(G,\Gamma)} \,,$$

for any $f \in \mathcal{H}(G,\Gamma)$. Let $\pi_\alpha \widetilde{\times (1 \otimes} \rho)$ denote the extension of $\pi_\alpha \times (1 \otimes \rho)$ to $M(C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)$, which is faithful since $\pi_\alpha \times (1 \otimes \rho)$ is faithful on $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. We have that $\pi_\alpha \widetilde{\times (1 \otimes} \rho)$ and $(1 \otimes \rho)$ coincide in $\mathcal{H}(G,\Gamma)$ since they are given by the same expression on the dense subspace $[\pi_\alpha \times (1 \otimes \rho)](C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H}$. Thus, we have

$$[\pi_\alpha \widetilde{\times (1 \otimes} \rho)](f) = (1 \otimes \rho)(f) \,,$$

for any $f \in \mathcal{H}(G,\Gamma)$. It then follows that

$$\begin{aligned} \|f\|_{M(C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)} &= \|[\pi_\alpha \widetilde{\times (1 \otimes} \rho)](f)\| = \|(1 \otimes \rho)(f)\| \\ &= \|\rho(f)\| = \|f\|_{C_r^*(G,\Gamma)} \,. \end{aligned}$$

This finishes the proof. $\square$

As it is known, reduced C^*-crossed products by discrete groups satisfy a universal property among all the C^*-completions of the *-algebraic crossed product that have a certain conditional expectation. This universal property says that every such completion has a canonical surjective map onto the reduced C^*-crossed product. As a consequence, the reduced C^*-crossed product is the only C^*-completion of the *-algebraic crossed product that has a certain faithful conditional expectation.

The next result explains how this holds in the Hecke pair case.

Theorem 8.2.11. *Let $\|\cdot\|_\tau$ be an α-extendable C^*-norm on $D(\mathcal{A})$ and $\|\cdot\|_\omega$ a C^*-norm on $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ whose restriction to $C_c(\mathcal{A}/\Gamma)$ is just the norm $\|\cdot\|_\tau$. Let us denote by $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,\omega} G/\Gamma$ the completion of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ under the norm $\|\cdot\|_\omega$.*

If there exists a bounded linear map $F : C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,\omega} G/\Gamma \to C^*_\tau(\mathcal{A}/\Gamma)$ such that

$$F(f) = f(\Gamma)\,,$$

for all $f \in C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$, then:

a) there exists a surjective *-homomorphism

$$\Lambda : C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,\omega} G/\Gamma \to C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,r} G/\Gamma\,,$$

such that Λ is the identity on $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$.

b) F is a conditional expectation.

c) F is faithful if and only if Λ is an isomorphism.

Proof: Let X_0 be the space $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$. It is easily seen that X_0 is a (right) inner product $C_c(\mathcal{A}/\Gamma)$-module, where $C_c(\mathcal{A}/\Gamma)$ acts on X_0 by right multiplication and the inner product is given by

$$\langle f_1\,,\, f_2\rangle := (f_1^* * f_2)(\Gamma)\,.$$

Since for any $f \in X_0$ and $f_1 \in C_c(\mathcal{A}/\Gamma)$ we have

$$\begin{aligned}\|\langle f * f_1\,,\, f * f_1\rangle\|_\tau &= \|((f*f_1)^* * (f*f_1))(\Gamma)\|_\tau\\ &= \|(f_1^* * f^* * f * f_1)(\Gamma)\|_\tau\\ &= \|f_1^*((f^* * f)(\Gamma))f_1\|_\tau\\ &= \|f_1\|_\tau^2\|\langle f\,,\, f\rangle\|_\tau\,,\end{aligned}$$

it follows that we can complete X_0 to a (right) Hilbert $C^*_\tau(\mathcal{A}/\Gamma)$-module, which we will denote by X. The inner product on X, which extends the inner product $\langle \cdot\,,\,\cdot\rangle$ above, will be denoted by $\langle \cdot\,,\,\cdot\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}$.

The *-algebra $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ acts on X_0 by left multiplication and therefore it is easily seen that this action is compatible with the right module structure. Moreover, $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ acts on X_0 by bounded operators, relatively to the norm induced by the inner product $\langle \cdot\,,\,\cdot\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}$, as we now show. For this we recall the conditional expectation E_Γ of $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,r} G/\Gamma$ onto $C^*_\tau(\mathcal{A}/\Gamma)$ as defined in Proposition 8.2.6. For any $f, f_1 \in C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ we have that inside $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,r} G/\Gamma$ the following holds:

$$\begin{aligned}
\langle f * f_1\,,\, f*f_1\rangle_{C^*_\tau(\mathcal{A}/\Gamma)} &= ((f*f_1)^* * (f*f_1))(\Gamma)\\
&= E_\Gamma((f*f_1)^* * (f*f_1))\\
&= E_\Gamma(f_1^* * f^* * f * f_1)\\
&\leq \|f\|^2_{\tau,r} E_\Gamma(f_1^* * f_1)\\
&= \|f\|^2_{\tau,r}\langle f_1\,,\, f_1\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\,,
\end{aligned}$$

where we used the positivity of E_Γ in $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,r} G/\Gamma$. Since the norm $\|\cdot\|_\tau$ is just the restriction of the norm $\|\cdot\|_{\tau,r}$ we get

$$\|\langle f * f_1\,,\, f*f_1\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau \;\leq\; \|f\|^2_{\tau,r}\|\langle f_1\,,\, f_1\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau\,, \qquad (8.9)$$

which shows that $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ acts on X_0 by bounded operators. Moreover, inequality (8.9) shows that this action extends to an action of $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,r} G/\Gamma$ on X and thus gives rise to a *-homomorphism from $\Phi : C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,r} G/\Gamma \to \mathcal{L}(X)$. We will now show that Φ is injective.

As usual, $Y := C^*_\tau(\mathcal{A}/\Gamma)$ is a Hilbert module over itself. We define the map $j_\Gamma : Y \to X$ simply by inclusion, i.e. $j_\Gamma(f) := f$. It is then easy to see that j_Γ is adjointable with adjoint $j^*_\Gamma : X \to Y$ given by $j_\Gamma(f) = f(\Gamma)$, for any $f \in X_0$. It is also easy to see that, for any $f \in C_c(\mathcal{A}/\Gamma)$ we have

$$\langle j_\Gamma(f)\,,\, j_\Gamma(f)\rangle_{C^*_\tau(\mathcal{A}/\Gamma)} \;=\; \langle f\,,\, f\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\,,$$

where the inner product on the left (respectively, right) hand side corresponds to the inner product in X (respectively, in Y). Thus, j_Γ is an isometry between Y and X and has therefore norm 1.

Let $\widehat{E} : \Phi(C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma) \to C^*_\tau(\mathcal{A}/\Gamma)$ be the map defined by

$$\widehat{E}(\Phi(f)) := \Phi(f(\Gamma))\,.$$

We claim that $\widehat{E}$ is continuous with respect to the norm of $\mathcal{L}(X)$. First we notice that for any $f \in C_c(\mathcal{A}/\Gamma)$ we have that (as elements of $\mathcal{L}(Y)$)

$$f = j^*_\Gamma \Phi(f) j_\Gamma\,.$$

Let $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. We have

$$\|\widehat{E}(\Phi(f))\|_{\mathcal{L}(X)} = \|\Phi(f(\Gamma))\|_{\mathcal{L}(X)}\,.$$

Since $\mathcal{L}(X)$ is a C^*-algebra, the norm $\|\cdot\|_{\mathcal{L}(X)}$ when restricted to $\Phi(C_c(\mathcal{A}/\Gamma))$ is such that $\|\Phi(g)\|_{\mathcal{L}(X)} = \|g\|_\tau$, and moreover the norm $\|\cdot\|_\tau$ coincides with the norm $\|\cdot\|_{\mathcal{L}(Y)}$, since $\mathcal{L}(Y) = M(C^*_\tau(\mathcal{A}/\Gamma))$. Hence we have:

$$\begin{array}{rcccl}\|\widehat{E}(\Phi(f))\|_{\mathcal{L}(X)} & = & \|\Phi(f(\Gamma))\|_{\mathcal{L}(X)} & = & \|f(\Gamma)\|_{\mathcal{L}(Y)} \\ & = & \|j^*_\Gamma\Phi(f)j_\Gamma\|_{\mathcal{L}(Y)} & \leq & \|\Phi(f)\|_{\mathcal{L}(X)}\,,\end{array}$$

which shows that $\widehat{E}$ is continuous with respect to the norm of $\mathcal{L}(X)$.

We can now prove that Φ is injective. First we notice that for any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have $\widehat{E}(\Phi(f)) = \Phi(E_\Gamma(f))$. By continuity, this equality then holds for any $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Suppose now that $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ is such that $\Phi(f) = 0$. Then we have

$$0 \;=\; \widehat{E}(\Phi(f^* * f)) \;=\; \Phi(E_\Gamma(f^* * f))\,.$$

Since Φ is faithful on $C^*_\tau(\mathcal{A}/\Gamma)$, it then follows that $E_\Gamma(f^* * f) = 0$, and since E_Γ is faithful this implies that $f^* * f = 0$, i.e. $f = 0$. Thus, Φ is injective.

We now want to prove part $a)$ of the theorem and for that we need to show that F satisfies certain properties. Let $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. We have that

$$\begin{array}{rcl}F(f^* * f) & = & (f^* * f)(\Gamma) \\ & = & \displaystyle\sum_{[h]\in G/\Gamma} f^*(h\Gamma)\alpha_h(f(h^{-1}\Gamma)) \\ & = & \displaystyle\sum_{[h]\in G/\Gamma} \Delta(h^{-1})\,\alpha_h(f(h^{-1}\Gamma))^*\alpha_h(f(h^{-1}\Gamma))\,,\end{array}$$

from which it follows that F is positive. Moreover, for $f_1 \in C_c(\mathcal{A}/\Gamma)$ we have that

$$\begin{array}{rcccl}F(f_1 * f) & = & (f_1 * f)(\Gamma) & = & f_1 \cdot f(\Gamma) \\ & = & f_1 \cdot F(f)\,, & & \end{array}$$

and similarly $F(f * f_1) = F(f) \cdot f_1$. By continuity of F, it follows that $F(f_1 * f) = f_1 \cdot F(f)$ and $F(f * f_1) = F(f) \cdot f_1$ for any $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ and $f_1 \in C^*_\tau(\mathcal{A}/\Gamma)$. Hence, apart form contractivity, we have shown that F satisfies all the other requirements for it to be a conditional expectation.

Let $f, g \in C_c(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$. We have that inside $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ the following holds:

$$\begin{aligned}
\langle f * g \,,\, f * g \rangle_{C_\tau^*(\mathcal{A}/\Gamma)} &= ((f * g)^* * (f * g))(\Gamma) \\
&= F((f * g)^* * (f * g)) \\
&= F(g^* * f^* * f * g) \\
&\leq \|f\|_\omega^2 F(g^* * g) \\
&= \|f\|_\omega^2 \langle g \,,\, g \rangle_{C_\tau^*(\mathcal{A}/\Gamma)} \,,
\end{aligned}$$

where we have used to the positivity of F. Since the norm $\|\cdot\|_\tau$ is just the restriction of the norm $\|\cdot\|_\omega$ we get

$$\|\langle f * g \,,\, f * g \rangle_{C_\tau^*(\mathcal{A}/\Gamma)}\|_\tau \;\leq\; \|f\|_\omega^2 \|\langle g \,,\, g \rangle_{C_\tau^*(\mathcal{A}/\Gamma)}\|_\tau \,, \tag{8.10}$$

which shows that the action of $C_c(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ on X_0 extends to an action of $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ on X and thus gives rise to a *-homomorphism from $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ to $\mathcal{L}(X)$. As the injectivity of Φ shows, the closure of the image of $C_c(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ in $\mathcal{L}(X)$ is isomorphic to $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Hence, we conclude that there is a map $\Lambda : C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma \to C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ such that $\Lambda(f) = f$, for $f \in C_c(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$.

To prove b) it remains to see that F is a contraction, and that just follows from the fact $F = E_\Gamma \circ \Lambda$.

Let us now prove c). The direction ($\Longleftarrow$) is clear, because F is then nothing but the conditional expectation E_Γ, which is faithful. Let us now prove the direction ($\Longrightarrow$). For any $f \in C_c(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ we have that $E_\Gamma \circ \Lambda(f^* * f) = F(f^* * f)$. By continuity this formula holds for any $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$. Let $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ be such that $\Lambda(f) = 0$. Then we necessarily have that $0 = E_\Gamma \circ \Lambda(f^* * f) = F(f^* * f)$, and since F is faithful we have that $f^* * f = 0$, i.e. $f = 0$. $\square$

8.3 Alternative definition of $C_r^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$

The C^*-direct limit $D_r(\mathcal{A})$ played a key role in the definition of the reduced crossed product $C_r^*(\mathcal{A}/\Gamma) \rtimes_{\alpha,r} G/\Gamma$. In this section we will see that instead of $D_r(\mathcal{A})$ one can use the more natural C^*-algebra $M(C_r^*(\mathcal{A}))$ to define the reduced crossed product $C_r^*(\mathcal{A}/\Gamma) \rtimes_{\alpha,r} G/\Gamma$. The algebra $M(C_r^*(\mathcal{A}))$ has several advantages over $D_r(\mathcal{A})$. For instance $C_r^*(\mathcal{A})$ appears more naturally in the setup for defining crossed products (recall that we start with the bundle $\mathcal{A}$ and then we form the various bundles $\mathcal{A}/\Gamma^g$ from it). Also, $C_r^*(\mathcal{A})$, being a cross sectional algebra of a Fell bundle, seems to be structurally simpler than $D_r(\mathcal{A})$, which is a direct limit of cross sectional algebras over Fell bundles.

The question one might ask at this point is: can one similarly use $M(C^*(\mathcal{A}))$ instead of $D_{\max}(\mathcal{A})$ in order to define $C^*(\mathcal{A}/\Gamma) \rtimes_{\alpha,r} G/\Gamma$? As we shall also see in this section, this is not possible in general. At the core of this problem lies the fact that one has always an embedding

$$C_r^*(\mathcal{A}/H) \to M(C_r^*(\mathcal{A}))\,,$$

extending the natural embedding of $C_c(\mathcal{A}/H)$ into $M(C_c(\mathcal{A}))$, whereas the analogous map

$$C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))\,,$$

is not always injective. This implies that the while the algebra $D_r(\mathcal{A})$ embeds naturally in $M(C_r(\mathcal{A}))$, the analogous map from $D_{\max}(\mathcal{A})$ to $M(C^*(\mathcal{A}))$ is not an embedding in general.

We start with the following general result:

Proposition 8.3.1. *Let $\|\cdot\|_\tau$ be any C^*-norm on $C_c(\mathcal{A})$ and $C_\tau^*(\mathcal{A})$ its completion. There is a unique mapping $C^*(\mathcal{A}/H) \to M(C_\tau^*(\mathcal{A}))$ which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$.*

Proof: As is known $C_\tau^*(\mathcal{A})$ is naturally a Hilbert $C_\tau^*(\mathcal{A})$-module, whose algebra of adjointable operators $\mathcal{L}(C_\tau^*(\mathcal{A}))$ is precisely the multiplier algebra $M(C_\tau^*(\mathcal{A}))$. In particular $X := C_c(\mathcal{A})$ is an inner product $C_\tau^*(\mathcal{A})$-module. Moreover, X is also a right $C_c(\mathcal{A}) - C_\tau^*(\mathcal{A})$ bimodule and a right $C_c(\mathcal{A}/H) - C_\tau^*(\mathcal{A})$ bimodule (in the sense of Definition 8.1.2), under the canonical actions of $C_c(\mathcal{A})$ and $C_c(\mathcal{A}/H)$ on X. Since $C_c(\mathcal{A})$ acts on X by bounded operators, it then follows from Lemma 8.1.3 (taking $K = \{e\}$) that $C_c(\mathcal{A}/H)$ acts on X by bounded operators. Thus, by completion, we obtain a right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}C_\tau^*(\mathcal{A})_{C_\tau^*(\mathcal{A})}$. Hence obtain a unique map $C^*(\mathcal{A}/H) \to M(C_\tau^*(\mathcal{A}))$ which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$. $\square$

As shall see later in this section the map $C^*(\mathcal{A}/H) \to M(C_\tau^*(\mathcal{A}))$ is not an embedding in general, not even when $C_\tau^*(\mathcal{A}) = C^*(\mathcal{A})$. Nevertheless for the reduced norms we have the following result:

Theorem 8.3.2. *There is a unique embedding of $C_r^*(\mathcal{A}/H)$ into $M(C_r^*(\mathcal{A}))$ which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$.*

Proof: From Proposition 8.3.1 we know that there exists a unique *-homo-morphism of $C^*(\mathcal{A}/H)$ to $M(C_r^*(\mathcal{A}))$, which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$. Thus, we have a right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}C_r^*(\mathcal{A})_{C_r^*(\mathcal{A})}$. Taking the balanced tensor product of this right-Hilbert bimodule together with ${}_{C_r^*(\mathcal{A})}L^2(\mathcal{A})_{C_0(\mathcal{A}^0)}$ we get a $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0)$ right-Hilbert bimodule

$$_{C^*(\mathcal{A}/H)}\Big(C_r^*(\mathcal{A}) \otimes_{C_r^*(\mathcal{A})} L^2(\mathcal{A})\Big)_{C_0(\mathcal{A}^0)}.$$

Since the action of $C_r^*(\mathcal{A})$ on $L^2(\mathcal{A})$ is faithful, the kernels of the maps from $C^*(\mathcal{A}/H)$ to $M(C_r^*(\mathcal{A}))$ and $\mathcal{L}\Big(C_r^*(\mathcal{A}) \otimes_{C_r^*(\mathcal{A})} L^2(\mathcal{A})\Big)$ are the same.

Now, $C_r^*(\mathcal{A}) \otimes_{C_r^*(\mathcal{A})} L^2(\mathcal{A})$ is isomorphic to $L^2(\mathcal{A})$ as a Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0)$ bimodule. Hence, it follows that the kernel of the map from $C^*(\mathcal{A}/H)$ to $M(C_r^*(\mathcal{A}))$ is the same as the kernel of the map from $C^*(\mathcal{A}/H)$ to $\mathcal{L}(L^2(\mathcal{A}))$. Now, the latter map has the same kernel as the canonical map $\Lambda : C^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/H)$, by Lemma 8.1.4 applied when K is the trivial subgroup. Thus, this gives an embedding of $C_r^*(\mathcal{A}/H)$ into $M(C_r^*(\mathcal{A}))$. $\square$

The next result is a generalization of [10, Proposition 2.10] (see Example 5.2.4). Its proof relies ultimately on Lemma 8.1.4, whose proof, we recall, was essentially an adaptation of the proof [10, Proposition 2.10] itself.

Corollary 8.3.3. *Suppose $\mathcal{A}$ is amenable. Then, the kernel of the canonical map $C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))$ is the same as the kernel of the canonical map $\Lambda : C^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/H)$.*

Proof: In the proof of Proposition 8.3.2 we established that the kernel of the canonnical map $\Lambda : C^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/H)$ is the same as the kernel of the map $C^*(\mathcal{A}/H) \to M(C_r^*(\mathcal{A}))$, which is the same as the map $C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))$ by amenability of $\mathcal{A}$. $\square$

We now give an example where the map $C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))$ is not injective:

Example 8.3.4. Let $\mathcal{B}$ be a non-amenable Fell bundle over the group G, and let $\mathcal{A} := \mathcal{B} \times G$ be the associated Fell bundle over the transformation groupoid $G \times G$. Following Example 5.2.4, we have a right G-action on the groupoid $G \times G$, given by $(s,t)g := (s,tg)$, and $\mathcal{A}$ has G-invariant fibers. Moreover, since the G-action is free, it is H-good and satisfies the H-intersection property, for any subgroup $H \subseteq G$. In this example we will consider H to be the whole group G. In this case the orbit groupoid $(G \times G)/G$ can be naturally identified with the group G, and moreover, the Fell bundle $\mathcal{A}/G$ is naturally identified with $\mathcal{B}$.

It is known that the bundle $\mathcal{A}$ is always amenable (see [10, Remark 2.11]), and therefore by Corollary 8.3.3 we have that the kernel of the map $C^*(\mathcal{A}/G) \to M(C^*(\mathcal{A}))$ is the same as the kernel of the canonical map $C^*(\mathcal{A}/G) \to C^*_r(\mathcal{A}/G)$. As we pointed out above, the bundle $\mathcal{A}/G$ is just $\mathcal{B}$, which is non-amenable by assumption. Hence, the canonical map $C^*(\mathcal{A}/G) \to C^*_r(\mathcal{A}/G)$ has a non-trivial kernel, and therefore the map $C^*(\mathcal{A}/G) \to M(C^*(\mathcal{A}))$ is not injective.

We will now see that $D_r(\mathcal{A})$ is canonically embedded in $M(C^*_r(\mathcal{A}))$, being the C^*-algebra generated by all the images of $C^*_r(\mathcal{A}/H)$ inside $M(C^*_r(\mathcal{A}))$, as in Proposition 8.3.2, with $H \in \mathcal{C}$.

Proposition 8.3.5. *Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. Then, the following diagram of canonical embeddings commutes:*

$$\begin{tikzcd} C^*_r(\mathcal{A}/H) \arrow[r] \arrow[dr] & C^*_r(\mathcal{A}/K) \arrow[d] \\ & M(C^*_r(\mathcal{A})) . \end{tikzcd} \tag{8.11}$$

As a consequence $D_r(\mathcal{A})$ embedds in $M(C^*_r(\mathcal{A}))$, being *-isomorphic to the subalgebra of $M(C^*_r(\mathcal{A}))$ generated by all the $C^*_r(\mathcal{A}/H)$, with $H \in \mathcal{C}$.

Proof: We have already proven in Proposition 7.1.4 that

$$a_{xH} b_y = \sum_{[h] \in \mathcal{S}_x \backslash H/K} a_{xhK} b_y \,, \tag{8.12}$$

for any $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$. Hence, by linearity, density and continuity, we conclude that diagram (8.11) commutes. By the universal property of $D_r(\mathcal{A})$ we then have a *-homomorphism from $D_r(\mathcal{A})$ to $M(C^*_r(\mathcal{A}))$, whose image is generated by all the images of $C^*_r(\mathcal{A}/H)$ inside $M(C^*_r(\mathcal{A}))$, for any $H \in \mathcal{C}$. This *-homomorphism from $D_r(\mathcal{A})$ to $M(C^*_r(\mathcal{A}))$ is injective because all the maps in diagram (8.11) are injective. $\square$

We can now give an equivalent definition for the reduced crossed product $C^*_r(\mathcal{A}/\Gamma) \times_{r,\alpha} G/\Gamma$, using the algebra $C^*_r(\mathcal{A})$ instead of $D^*_r(\mathcal{A})$. This can be advantageous as we observed in the opening paragraph of this section. Also, this equivalence of definitions makes the connection between our definition of a reduced crossed product by a Hecke pair and that of Laca, Larsen and Neshveyev in [30], as we shall see in the next section.

Theorem 8.3.6. *Let $\pi : C^*_r(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation, and $\widetilde{\pi}$ its extension to $M(C^*_r(\mathcal{A}))$. We have that*

i) If $\widetilde{\pi}_\alpha : C^*_r(\mathcal{A}/\Gamma) \to B(\mathcal{H} \otimes \ell^2(G/\Gamma))$ is faithful, then $\widetilde{\pi}_\alpha \times (1 \otimes \rho)$ is a faithful representation of $C^*_r(\mathcal{A}/\Gamma) \times_{r,\alpha} G/\Gamma$. Consequently,

$$\|f\|_{r,r} := \|[\widetilde{\pi}_\alpha \times (1 \otimes \rho)](f)\|\,,$$

for all $f \in C^*_r(\mathcal{A}/\Gamma) \times_{r,\alpha} G/\Gamma$.

ii) If π is faithful, then $\widetilde{\pi}_\alpha$ is faithful.

Proof: By Proposition 8.3.5 $D_r(\mathcal{A})$ is canonically embedded in $M(C^*_r(\mathcal{A}))$, so that $\widetilde{\pi}$ restricts to a *-representation of $D_r(\mathcal{A})$. This restriction is nondegenerate, because the restriction to $C^*_r(\mathcal{A}/\Gamma)$ is already nondegenerate, as follows from the following argument. Let $\xi \in \mathscr{H}$ be such that $\widetilde{\pi}(C^*_r(\mathcal{A}/\Gamma))\xi = 0$. For any $x \in X$ and $a \in \mathcal{A}_x$ we have

$$\begin{aligned} \|\pi(a_x)\xi\|^2 &= \langle \pi((a^*a)_{\mathbf{s}(x)})\xi\,,\,\xi\rangle \\ &= \langle \pi(a^*_{x^{-1}} \cdot a_{x\Gamma})\xi\,,\,\xi\rangle \\ &= \langle \pi(a^*_{x^{-1}})\widetilde{\pi}(a_{x\Gamma})\xi\,,\,\xi\rangle \\ &= 0\,. \end{aligned}$$

Thus, by nondegeneracy of π we get that $\xi = 0$, and therefore $\widetilde{\pi}$ restricted to $C^*_r(\mathcal{A}/\Gamma)$, and hence also $D_r(\mathcal{A})$, is nondegenerate. We are now in the conditions of Theorem 8.2.9.

Claim *ii)* also follows from Theorem 8.2.9, given the fact that a faithfull nondegenerate *-representation of $C^*_r(\mathcal{A})$ extends faithfully to $M(C^*_r(\mathcal{A}))$. $\square$

8.4 Comparison with Laca-Larsen-Neshveyev construction

In [30], Laca, Larsen and Neshveyev, based on the work of Connes-Marcolli [8] and Tzanev [41], introduced an algebra which could be thought of as a reduced crossed product of an abelian algebra by an action of a Hecke pair.

The construction introduced by Laca, Larsen and Neshveyev was one of the motivations behind our definition of a crossed product by a Hecke pair. However, the setup for Laca, Larsen and Neshveyev's construction is slightly different from ours, being on one side more particular, as it only allows one to take a crossed product by an abelian algebra, but also more general, as the underlying space is not assumed in [30] to be discrete. We will show in this section that when both setups agree, our crossed product is canonically isomorphic to the crossed product of [30].

We will first briefly recall the setup and construction presented in [30, Section 1]. In order to make a coherent and more meanigful comparison between our construction and that of [30] we will have to make a few simple modifications in the latter. Essentially, we will consider right actions of G instead of left ones, and make the appropriate changes in the construction of [30] according to this.

Let G be a group acting on the right on a locally compact space X. Let $\Gamma \subseteq G$ be a Hecke subgroup and consider the (right) action of $\Gamma \times \Gamma$ on $X \times G$, given by:

$$(x, g)(\gamma_1, \gamma_2) := (x\gamma_1, \gamma_1^{-1} g\gamma_2)\,. \tag{8.13}$$

Define $X \times_\Gamma G/\Gamma$ to be the quotient space of $X \times G$ by the action of $\Gamma \times \Gamma$. We assume that the space $X \times_\Gamma G/\Gamma$ is Hausdorff.

Remark 8.4.1. In [30] the original assumption was that the action of Γ on X was proper (hence implying that $X \times_\Gamma G/\Gamma$ is Hausdorff), but as it was observed in [30, Remark 1.4], requiring that $X \times_\Gamma G/\Gamma$ is Hausdorff was actually enough for the construction to make sense, and this is an important detail for us as the actions we consider are not proper in general.

Let $C_c(X \times_\Gamma G/\Gamma)$ be the space of compactly supported continuous functions on $X \times_\Gamma G/\Gamma$. We will view the elements of $C_c(X \times_\Gamma G/\Gamma)$ as $(\Gamma \times \Gamma)$-invariant functions on $X \times G$. One can define a convolution product and involution in $C_c(X \times_\Gamma G/\Gamma)$ according to the following formulas:

$$(f_1 * f_2)(x, g) := \sum_{[h] \in G/\Gamma} f_1(x, h) f_2(xh, h^{-1}g)\,, \tag{8.14}$$

$$f^*(x, g) := \overline{f(xg, g^{-1})}\,. \tag{8.15}$$

For each given $x \in X$ we can define a *-representation $\pi_x : C_c(X \times_\Gamma G/\Gamma) \to$

$B(\ell^2(G/\Gamma))$ by

$$\pi_x(f)\delta_{h\Gamma} := \sum_{g\in G/\Gamma} f(xg, g^{-1}h)\delta_{g\Gamma}\,. \tag{8.16}$$

The C^*-algebra $C_r^*(X\times_\Gamma G/\Gamma)$ is defined as the completion of $C_c(X\times_\Gamma G/\Gamma)$ in the norm

$$\|f\| := \sup_{x\in X}\|\pi_x(f)\|\,. \tag{8.17}$$

The setup behind this construction differs slightly from our own, so we will compare both constructions under the following assumptions:

- (G,Γ) is a Hecke pair;
- X is a set (seen as both a discrete space and a discrete groupoid);
- There is a right action of G on X;
- The G-action satisfies the Γ-intersection property.

We notice that since X and G are discrete the space $X\times_\Gamma G/\Gamma$ is also discrete and therefore Hausdorff, so that the necessary assumptions for the construction of [30] are satisfied. Also, since X is just a set, the action G on X is necessarily Γ-good. Our "standing assumption" 6.0.11 is thus satisfied for the trivial Fell bundle $\mathcal{A}$ over X in which every fiber $\mathcal{A}_x$ is just $\mathbb{C}$. Recall that in this case $C_c(\mathcal{A}) = C_c(X)$ and $C_c(\mathcal{A}/\Gamma) = C_c(X/\Gamma)$.

Theorem 8.4.2. *Let (G,Γ) be a Hecke pair and X a set. Assume that there is a right G-action on X which satisfies the Γ-intersection property. Then, the map $\Phi : C_c(X/\Gamma)\times_\alpha^{alg} G/\Gamma \to C_c(X\times_\Gamma G/\Gamma)$ given by*

$$\Phi(f)\,(x,g) := \Delta(g)^{\frac{1}{2}}\, f(g\Gamma)(x\Gamma^g)\,,$$

is a *-isomorphism. This map extends to a *-isomorphism of the reduced completions $\Phi : C_0(X/\Gamma)\times_{\alpha,r} G/\Gamma \to C_r^*(X\times_\Gamma G/\Gamma)$. Moreover, under the *-isomorphism Φ, the *-representation π_x is just $(\widetilde{\varphi_x})_\alpha\times\rho$, where φ_x is the *-representation of $C_0(X)$ given by evaluation at x, i.e. $\varphi_x(f) = f(x)$.

Proof: Let us first check that Φ is well-defined, i.e. $\Phi(f)$ is a $(\Gamma \times \Gamma)$-invariant function in $G \times X$, with compact support (as a function on $X \times_\Gamma G/\Gamma$). To see this, let $\gamma_1, \gamma_2 \in \Gamma$. We have that

$$\begin{aligned}
\Phi(f)\,(x\gamma_1, \gamma_1^{-1} g\gamma_2) &= \Delta(\gamma_1^{-1} g\gamma_2)^{\frac{1}{2}}\, f(\gamma_1^{-1} g\gamma_2\Gamma)(x\gamma_1\Gamma^{\gamma_1^{-1} g\gamma_2}) \\
&= \Delta(g)^{\frac{1}{2}}\, \alpha_{\gamma_1^{-1}}(f(g\Gamma))\,(x\gamma_1\Gamma^{\gamma_1^{-1} g}) \\
&= \Delta(g)^{\frac{1}{2}}\, f(g\Gamma)\,(x\Gamma^{g}) \\
&= \Phi(f)\,(x, g)\,,
\end{aligned}$$

so that $\Phi(f)$ is $\Gamma \times \Gamma$-invariant. It is easy to see that $\Phi(f)$ has compact support (as a function on $X \times_\Gamma G/\Gamma$). Thus, Φ is well-defined.

Let us now prove that Φ is a *-homomorphism. It is clear that Φ is linear, so that we only need to check that Φ preserves products and the involution. For $f_1, f_2 \in C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have that

$$\begin{aligned}
&\Phi(f_1 * f_2)\,(x, g) \;= \\
= \;& \Delta(g)^{\frac{1}{2}}\,(f_1 * f_2)(g\Gamma)\,(x\Gamma^{g}) \\
= \;& \sum_{[h] \in G/\Gamma} \Delta(g)^{\frac{1}{2}}\, f_1(h\Gamma)\alpha_h(f_2(h^{-1}g\Gamma))\,(x\Gamma^{g}) \\
= \;& \sum_{[h] \in G/\Gamma} \Big(\Delta(h)^{\frac{1}{2}}\, f_1(h\Gamma)\,(x\Gamma^{h})\Big)\Big(\Delta(h^{-1}g)^{\frac{1}{2}}\,\alpha_h(f_2(h^{-1}g\Gamma))\,(x(g\Gamma g^{-1} \cap h\Gamma h^{-1}))\Big) \\
= \;& \sum_{[h] \in G/\Gamma} \Big(\Phi(f_1)\,(x, h)\Big)\Big(\Delta(h^{-1}g)^{\frac{1}{2}}\, f_2(h^{-1}g\Gamma)\,(xh\Gamma^{h^{-1}g})\Big) \\
= \;& \sum_{[h] \in G/\Gamma} \Big(\Phi(f_1)\,(x, h)\Big)\Big(\Phi(f_2)(xh, h^{-1}g)\Big) \\
= \;& \Phi(f_1) * \Phi(f_2)\,(x, g)\,.
\end{aligned}$$

Also for $f \in C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have

$$\begin{aligned}
\Phi(f^*)\,(x, g) &= \Delta(g)^{\frac{1}{2}}\, f^*(g\Gamma)\,(x\Gamma^{g}) \;=\; \Delta(g)^{\frac{1}{2}}\Delta(g^{-1})\,\overline{\alpha_g(f(g^{-1}))\,(x\Gamma^{g})} \\
&= \Delta(g^{-1})^{\frac{1}{2}}\,\overline{f(g^{-1})\,(xg\Gamma^{g^{-1}})} \;=\; \overline{\Phi(f)\,(xg, g^{-1})} \\
&= (\Phi(f))^*\,(x, g)\,.
\end{aligned}$$

Hence, Φ is a *-homomorphism. Let us now prove that Φ is injective. Suppose $\Phi(f) = 0$. Then for every $g \in G$ and $x \in X$ we have

$$0 \;=\; \Phi(f)\,(x, g) \;=\; \Delta(g)^{\frac{1}{2}}\, f(g\Gamma)\,(x\Gamma^{g})\,.$$

Hence, we conclude that $f(g\Gamma) = 0$ for all $g \in G$, and therefore $f = 0$, i.e. Φ is injective.

Let us now prove the surjectivity of Φ. The elements of $C_c(X \times_\Gamma G/\Gamma)$ are simply linear combinations of characteristic functions of elements of $X \times_\Gamma G/\Gamma$, so in order to prove that Φ is surjective we only need to check that each of these characteristic functions belongs the image of Φ. Let $[(x,g)] \in X \times_\Gamma G/\Gamma$. We claim that $\Phi(\Delta(g)^{-\frac{1}{2}}\, 1_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma}) = 1_{[(x,g)]}$. To see this, we recall Lemma 6.1.15 and notice that

$$\Phi(1_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma})\,(x,g) \quad = \quad \Delta(g)^{\frac{1}{2}}\,.$$

It is not difficult to see that $\Phi(1_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma})\,(y,h) = 0$ if (y,h) does not belong to the $\Gamma \times \Gamma$-orbit of (x,g), so that $\Phi(\Delta(g)^{-\frac{1}{2}}\, 1_{x\Gamma} * \Gamma g \Gamma * 1_{xg\Gamma}) = 1_{[(x,g)]}$. Hence, we can conclude that Φ is surjective and therefore establishes a *-isomorphism between $C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma$ and $C_c(X \times_\Gamma G/\Gamma)$.

We will now see that under the *-isomorphism Φ, the *-representation π_x is just $(\varphi_x)_\alpha \times \rho$, in other words $\pi_x \circ \Phi = (\varphi_x)_\alpha \times \rho$. This follows from the following computation:

$$\begin{aligned}
\pi_x \circ \Phi(f)\, \delta_{h\Gamma} &= \sum_{g\Gamma \in G/\Gamma} \Phi(f)(xg, g^{-1}h)\, \delta_{g\Gamma} \\
&= \sum_{g\Gamma \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, f(g^{-1}h\Gamma)(xg\Gamma^{g^{-1}h})\, \delta_{g\Gamma} \\
&= \sum_{g\Gamma \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \alpha_g(f(g^{-1}h\Gamma))(x(h\Gamma h^{-1} \cap g\Gamma g^{-1}))\, \delta_{g\Gamma} \\
&= \sum_{g\Gamma \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \widetilde{\varphi_x}\big(\alpha_g(f(g^{-1}h\Gamma))\big)\, \delta_{g\Gamma} \\
&= [(\widetilde{\varphi_x})_\alpha \times \rho](f)\, \delta_{g\Gamma}\,.
\end{aligned}$$

Let us now prove that the *-isomorphism Φ extends to a *-isomorphism between $C_0(X/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C_r^*(X \times_\Gamma G/\Gamma)$. Let $\pi : C_c(X \times_\Gamma G/\Gamma) \longrightarrow B(\ell^2(X))$ be the direct sum *-representation $\pi := \bigoplus_{x \in X} \pi_x$ on the Hilbert space $\bigoplus_{x\in X} \mathbb{C} \cong \ell^2(X)$. We then have that

$$\begin{aligned}
\pi \circ \Phi(f) &= \bigoplus_{x \in X} \pi_x(\Phi(f)) \\
&= \bigoplus_{x \in X} [(\widetilde{\varphi_x})_\alpha \times \rho](f) \\
&= [(\bigoplus_{x \in X} \widetilde{\varphi_x})_\alpha \times \rho](f) \\
&= [(\widetilde{\bigoplus_{x \in X} \varphi_x})_\alpha \times \rho](f)\,.
\end{aligned}$$

Now the *-representation $\bigoplus_{x\in X}\varphi_x$ of $C_0(X)$ is obviously injective. Hence, by Theorem 8.3.6 *ii*), it follows that $\pi\circ\Phi$ extends to a faithful *-representation of $C_0(X/\Gamma)\times_{\alpha,r}G/\Gamma$. This implies that Φ extends to an isomorphism between $C_0(X/\Gamma)\times_{\alpha,r}G/\Gamma$ and $C_r^*(X\times_\Gamma G/\Gamma)$, because

$$\begin{aligned}\|\Phi(f)\| &= \sup_{x\in X}\|\pi_x(\Phi(f))\| = \|\pi\circ\Phi(f)\| \\ &= \|f\|_{r,r}\,.\end{aligned}$$

$\square$

Chapter 9

Other completions

Just like there are several canonical C^*-completions of a Hecke algebra, one can consider also different C^*-completions for crossed products by Hecke pairs. Especially interesting for this work are full C^*-crossed products, but we will also take a look at C^*-completions arising from a L^1-norm.

9.1 Full C^*-crossed products

In this section we define and study full C^*-crossed products by Hecke pairs. Just like in the reduced case, several full C^*-crossed products can be considered, such as $C_r^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ where each of these is thought of as the full C^*-crossed product of $C_r^*(\mathcal{A}/\Gamma)$, respectively $C^*(\mathcal{A}/\Gamma)$, by the Hecke pair (G, Γ). As is the case for Hecke algebras, full crossed products by Hecke pairs do not have to exist in general.

Definition 9.1.1. Let $\| \cdot \|_\tau$ be an α-extendable C^*-norm in $D(\mathcal{A})$. We will denote by $\| \cdot \|_{\tau,u} : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \longrightarrow \mathbb{R}_0^+ \cup \{\infty\}$ the function defined by

$$\|f\|_{\tau,u} := \sup_{\Phi \in R_\tau} \|\Phi(f)\| \,, \tag{9.1}$$

where the supremum is taken over the class R_τ of *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ whose restrictions to $C_c(\mathcal{A}/\Gamma)$ are continuous with respect to $\| \cdot \|_\tau$.

Proposition 9.1.2. *We have that $\| \cdot \|_{\tau,u}$ is a C^*-norm in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ if and only if $\|f\|_{\tau,u} < \infty$ for all $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.*

Proof: $(\Longrightarrow)$: This direction is trivial since a norm must take values in $\mathbb{R}_0^+$.

$(\Longleftarrow)$: It is clear in this case that $\|\cdot\|_{\tau,u}$ defines a C^*-seminorm. To check that it is a true C^*-norm it is enough to find a faithful *-representation $\Phi \in R_\tau$. This is easy because since $\|\cdot\|_\tau$ is α-extendable we can take any nondegenerate faithful *-representation π of $D_\tau(\mathcal{A})$ and take $\Phi := \pi_\alpha \times (1 \otimes \rho)$, which is a faithful *-representation by Theorem 8.2.9. We have that $\Phi \in R_\tau$ because its restriction to $C_c(\mathcal{A}/\Gamma)$ is just π_α, which is continuous with respect to $\|\cdot\|_\tau$ by Lemma 8.2.3. $\square$

Definition 9.1.3. Let $\|\cdot\|_\tau$ be an α-extendable C^*-norm in $D(\mathcal{A})$. When $\|\cdot\|_{\tau,u}$ is a C^*-norm we will call it *the universal norm associated to* $\|\cdot\|_\tau$. The completion of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ with respect to this norm will be denoted by $C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ and referred to as the *full crossed product* of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G,Γ).

It is clear that $\|\cdot\|_{\tau,r} \leq \|\cdot\|_{\tau,u}$ so that the identity map on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ extends to a surjective *-homomorphism

$$C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma \longrightarrow C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma\,, \tag{9.2}$$

in case $\|\cdot\|_{\tau,u}$ is a norm.

In general, full crossed products do not necessarily exist, as it is already clear from the fact that a Hecke algebra (which is a particular case of crossed product by a Hecke pair) does not need to have an enveloping C^*-algebra. Nevertheless, for Hecke pairs whose Hecke algebras are BG^*-algebra one can always assure the existence of full C^*-crossed products:

Theorem 9.1.4. *If $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra, then the full crossed product $C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ always exists, for any α-extendable norm $\|\cdot\|_\tau$.*

Proof: We will prove that when $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra we have

$$\sup_\Phi \|\Phi(f)\| < \infty\,, \tag{9.3}$$

where the supremum runs over the class of all *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. To see this we first notice that it is enough to consider nondegenerate *-representations. Secondly, from Theorem 6.2.15, any nondegenerate *-representation Φ of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is the integrated form of a covariant pre-*-representation $(\Phi|, \mu_\Phi)$, so that we can write $\Phi = \Phi| \times \mu_\Phi$. Taking any element $a_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$ of the canonical spanning set of elements of the crossed product we then have

$$\begin{aligned} \|\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\| &= \|\Phi|(a_{x\Gamma})\mu_\Phi(\Gamma g\Gamma)\widetilde{\Phi}|(1_{\mathbf{s}(x)g\Gamma})\| \\ &\leq \|\Phi|(a_{x\Gamma})\mu_\Phi(\Gamma g\Gamma)\| \,. \end{aligned}$$

Now, since $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra we have that μ_Φ is normed, i.e. $\mu_\Phi(\Gamma g\Gamma)$ is a bounded operator. Moreover, because it is a BG^*-algebra, $\mathcal{H}(G,\Gamma)$ has an enveloping C^*-algebra. Hence, we conclude that

$$\begin{aligned} &\leq \|\Phi|(a_{x\Gamma})\|\|\mu_\Phi(\Gamma g\Gamma)\| \\ &\leq \|a_{x\Gamma}\|_{C^*(\mathcal{A}/\Gamma)}\|\Gamma g\Gamma\|_{C^*(G,\Gamma)} \,. \end{aligned}$$

Thus, it is clear that

$$\sup_\Phi \|\Phi(a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\| < \infty \,.$$

Since this is true for the elements of the canonical spanning set, it follows that (9.3) holds for any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. $\square$

Any BG^*-algebra necessarily has an enveloping C^*-algebra. Is it then possible to weaken the assumptions on Theorem 9.1.4 to cover all Hecke algebras with an enveloping C^*-algebra? In other words:

Open Question 9.1.5. If $\mathcal{H}(G,\Gamma)$ has an enveloping C^*-algebra, do the full crossed products $C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ always exist?

We do not know the answer to this question. In fact we do not even have an example of a Hecke algebra which has an enveloping C^*-algebra and is not a BG^*-algebra. More generally even, the author does not know any example of a *-algebra that can be faithfully represented on a Hilbert space and has an enveloping C^*-algebra, but is not a BG^*-algebra.

Regarding the existence of full crossed products we will show, in the next chapter, that they can exist for Hecke pairs for which the Hecke algebra does not have an enveloping C^*-algebra. Namely, the full crossed product $C_0(G/\Gamma) \times_\alpha G/\Gamma$, arising from the action of G on itself by translation, exists for all Hecke pairs (G,Γ).

9.2 L^1-norm and associated C^*-completion

We now define a L^1-norm on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, whose corresponding enveloping C^*-algebra can still be understood as a crossed product of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G,Γ), for a α-extendable norm $\|\cdot\|_\tau$.

Definition 9.2.1. Let $\|\cdot\|_\tau$ be an α-extendable C^*-norm on $D(\mathcal{A})$. We define the norm $\|\cdot\|_{\tau,L^1}$ in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ by:

$$\|f\|_{\tau,L^1} := \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\,\|f(g\Gamma)\|_\tau\,. \tag{9.4}$$

Before we prove that $\|\cdot\|_{\tau,L^1}$ is a norm we observe that $\|\cdot\|_{\tau,L^1}$ is well-defined, i.e. it does not depend on the chosen representative g of $[g]$, because for any $\gamma\in\Gamma$ we have, using the fact that the $\|\cdot\|_\tau$ is α-extendable,

$$\|f(\gamma g\Gamma)\|_\tau \;=\; \|\alpha_\gamma(f(g\Gamma))\|_\tau \;=\; \|f(g\Gamma)\|_\tau\,.$$

With this observation at hand we can easily derive another formula for $\|\cdot\|_{\tau,L^1}$, for which we have

$$\|f\|_{\tau,L^1} = \sum_{[g]\in G/\Gamma} \|f(g\Gamma)\|_\tau\,. \tag{9.5}$$

Proposition 9.2.2. *The function $\|\cdot\|_{\tau,L^1}$ is a norm for which*

$$\|f_1 * f_2\|_{\tau,L^1} \le \|f_1\|_{\tau,L^1}\|f_2\|_{\tau,L^1} \qquad \text{and} \qquad \|f^*\|_{\tau,L^1} = \|f\|_{\tau,L^1}\,.$$

Thus, under this norm $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ becomes a normed *-algebra.

Proof: It is easy to check that $\|\cdot\|_{\tau,L^1}$ is a vector space norm in $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$. Let us prove first that $\|f^*\|_{\tau,L^1} = \|f\|_{\tau,L^1}$. We have

$$\begin{aligned}
\|f^*\|_{\tau,L^1} &= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\,\|f^*(g\Gamma)\|_\tau \\
&= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\Delta(g^{-1})\,\|\alpha_g(f(g^{-1}\Gamma))^*\|_\tau \\
&= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g^{-1})\,\|f(g^{-1}\Gamma)\|_\tau\,.
\end{aligned}$$

Since $[g] \mapsto [g^{-1}]$ is a bijection of the set $\Gamma\backslash G/\Gamma$, we get

$$\begin{aligned} &= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\, \|f(g\Gamma)\|_\tau \\ &= \|f\|_{\tau,L^1}\,. \end{aligned}$$

Let us now prove that $\|f_1 * f_2\|_{\tau,L^1} \leq \|f_1\|_{\tau,L^1}\|f_2\|_{\tau,L^1}$. For this we will use the formula for $\|\cdot\|_{\tau,L^1}$ given by (9.5). We have that

$$\begin{aligned} \|f_1 * f_2\|_{\tau,L^1} &= \sum_{[g]\in G/\Gamma} \|(f_1 * f_2)(g\Gamma)\|_\tau \\ &\leq \sum_{[g]\in G/\Gamma}\sum_{[h]\in G/\Gamma} \|f_1(h\Gamma)\|_\tau \|\alpha_h(f_2(h^{-1}g\Gamma))\|_\tau\,. \end{aligned}$$

Using the fact that $\|\cdot\|_\tau$ is α-extendable we have

$$\begin{aligned} &= \sum_{[g]\in G/\Gamma}\sum_{[h]\in G/\Gamma} \|f_1(h\Gamma)\|_\tau \|f_2(h^{-1}g\Gamma)\|_\tau \\ &= \sum_{[h]\in G/\Gamma}\sum_{[g]\in G/\Gamma} \|f_1(h\Gamma)\|_\tau \|f_2(h^{-1}g\Gamma)\|_\tau \\ &= \sum_{[h]\in G/\Gamma}\sum_{[g]\in G/\Gamma} \|f_1(h\Gamma)\|_\tau \|f_2(g\Gamma)\|_\tau \\ &= \Big(\sum_{[h]\in G/\Gamma} \|f_1(h\Gamma)\|_\tau\Big)\Big(\sum_{[g]\in G/\Gamma} \|f_2(g\Gamma)\|_\tau\Big) \\ &= \|f_1\|_{\tau,L^1}\|f_2\|_{\tau,L^1}\,. \end{aligned}$$

□

Completing $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ in the norm $\|\cdot\|_{\tau,L^1}$ we obtain a Banach *-algebra, and taking the enveloping C^*-algebra of this Banach *-algebra we obtain a C^*-completion of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$, which we denote by $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,L^1} G/\Gamma$. We notice that the restriction of the norm $\|\cdot\|_{\tau,L^1}$ to $C_c(\mathcal{A}/\Gamma)$ is precisely the norm $\|\cdot\|_\tau$, from which we can conclude that $\|\cdot\|_{\tau,u}$ is always greater or equal to the C^*-norm of $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,L^1} G/\Gamma$. This means that, if $\|\cdot\|_{\tau,u}$ is a norm, there is canonical map

$$C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma \to C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,L^1} G/\Gamma\,. \tag{9.6}$$

In case the crossed product is just the Hecke algebra itself, the map (9.6) is just the usual map

$$C^*(G,\Gamma) \to C^*(L^1(G,\Gamma))\,.$$

So far we have seen three canonical C^*-crossed products of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ), and these are $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha, r} G/\Gamma$, $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha, L^1} G/\Gamma$ and $C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ if it exists. Each one of these corresponds respectively, in the Hecke algebra case, to the completions $C^*_r(G, \Gamma)$, $C^*(L^1(G, \Gamma))$ and $C^*(G, \Gamma)$. It is an interesting problem, which we will not explore here, to understand how the Schlichting completion construction and the remaining Hecke C^*-algebra $pC^*(\overline{G})p$ carry over to crossed products by Hecke pairs.

Chapter 10

Stone-von Neumann theorem for Hecke pairs

A modern version of the Stone-von Neumann theorem in the language of crossed products by groups (see [37, Theorem C.34]) states that

$$C_0(G) \times_\alpha G \;\cong\; C_0(G) \times_{\alpha,r} G \;\cong\; \mathcal{K}(\ell^2(G))\,.$$

More precisely, if α is the action of G on $C_0(G)$ by right translation, $M : C_0(G) \to B(\ell^2(G))$ the *-representation by pointwise multiplication and ρ the right regular representation of G on $\ell^2(G)$, then (M, ρ) is a covariant representation of the system $(C_0(G), G)$ and $M \times \rho$ is a faithful *-representation of $C_0(G) \times_\alpha G$ with range $\mathcal{K}(\ell^2(G))$.

It follows from this result that any covariant representation of $(C_0(G), G)$ is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ), since the algebra of compact operators has a trivial representation theory ([37, Remark C.35]).

The goal of this chapter is to show how the Stone-von Neumann theorem generalizes to the setting of Hecke pairs and their crossed products. In the process we recover an Huef, Kaliszewski and Raeburn's ([19]) notion of a *covariant pair* and their version of the Stone-von Neumann theorem for Hecke pairs, which did not make use of crossed products.

10.1 Stone-von Neumann theorem for Hecke pairs

In [19, Definition 1.1], an Huef, Kaliszewski and Raeburn introduced the notion of a *covariant pair* (π, μ) consisting of a nondegenerate *-representation

$\pi : C_0(G/\Gamma) \to B(\mathscr{H})$ and a unital *-representation $\mu : \mathcal{H}(G,\Gamma) \to B(\mathscr{H})$ satisfying

$$\mu(\Gamma g\Gamma)\pi(1_{x\Gamma})\mu(\Gamma s\Gamma) = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \pi(1_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\pi(1_{xv\Gamma})\,. \tag{10.1}$$

The basic example of a covariant pair, computed in [19, Example 1.5], is that of (M,ρ) where $M : C_0(G/\Gamma) \to B(\ell^2(G/\Gamma))$ is the *-representation by pointwise multiplication and ρ is the right regular representation of $\mathcal{H}(G,\Gamma)$. One should note that the definition of the right regular representation ρ used in [19] differs from ours, since in [19] the factor $\Delta^{\frac{1}{2}}$ is absent. Nevertheless, (M,ρ) is still a covariant pair in our definition of ρ.

It was proven in [19, Theorem 1.6] that all covariant pairs are unitarily equivalent to an amplification $(1\otimes M, 1\otimes\rho)$ of (M,ρ), which can be seen as an analogue for Hecke pairs of the Stone-von Neumann theorem. It should be noted that this result was proven without any crossed product construction behind it.

In the following we will prove a Stone-von Neumann theorem for Hecke pairs in the language of crossed products, stating that

$$C_0(G/\Gamma)\times G/\Gamma \;\cong\; C_0(G/\Gamma)\times_r G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma))\,.$$

We will also show that the covariant pairs of [19] coincide with our notion of a covariant *-representation and we will recover an Huef, Kaliszewski and Raeburn's version of the Stone-von Neumann theorem ([19, Theorem 1.6]) as a consequence of the above isomorphisms.

The case under consideration now is that when the groupoid X is the set G and $\mathcal{A}$ is the Fell bundle over (the set) G whose fibers are $\mathbb{C}$. In this case we have $C_c(\mathcal{A}) = C_c(G)$ and, naturally, $C_r^*(\mathcal{A}) = C^*(\mathcal{A}) = C_0(G)$. We consider the action of G on itself by right multiplication. Since this action is free, it is Γ-good and satisfies the Γ-intersection property. Moreover, $\mathcal{A}$ clearly has G-invariant fibers. The groupoid X/Γ is then nothing but orbit set G/Γ, and $C_c(\mathcal{A}/\Gamma) = C_c(G/\Gamma)$. Moreover, $C_r^*(\mathcal{A}/\Gamma) = C^*(\mathcal{A}/\Gamma) = C_0(G/\Gamma)$.

Proposition 10.1.1. *Let $T_{g\Gamma,h\Gamma} \in C_c(G/\Gamma)\times_\alpha^{alg} G/\Gamma$ be the element*

$$T_{g\Gamma,h\Gamma} := 1_{g\Gamma} * \Gamma g^{-1}h\Gamma * 1_{h\Gamma}\,.$$

Then $\{T_{g\Gamma,h\Gamma}\}_{g\Gamma,h\Gamma\in G/\Gamma}$ is a set of matrix units that span $C_c(G/\Gamma)\times_\alpha^{alg} G/\Gamma$.

Proof: It is clear that $T^*_{g\Gamma,h\Gamma} = T_{h\Gamma,g\Gamma}$. Let us now compute the product $T_{g\Gamma,h\Gamma} * T_{s\Gamma,t\Gamma}$. If $h\Gamma \neq s\Gamma$, then $T_{g\Gamma,h\Gamma} * T_{s\Gamma,t\Gamma} = 0$. In case $h\Gamma = s\Gamma$, we get

$$\begin{aligned} T_{g\Gamma,h\Gamma} * T_{h\Gamma,t\Gamma} &= 1_{g\Gamma} * \Gamma g^{-1} h\Gamma * 1_{h\Gamma} * \Gamma h^{-1} t\Gamma * 1_{t\Gamma} \\ &= 1_{g\Gamma} * \Big(\sum_{\substack{[u]\in\Gamma h^{-1}g\Gamma/\Gamma \\ [v]\in\Gamma h^{-1}t\Gamma/\Gamma}} 1_{hu\Gamma} * \Gamma u^{-1} v\Gamma * 1_{hv\Gamma} \Big) * 1_{t\Gamma} \,. \end{aligned}$$

Now for the product $1_{g\Gamma} 1_{hu\Gamma}$ to be non-zero, we must have $hu\Gamma = g\Gamma$, i.e. $u\Gamma = h^{-1}g\Gamma$. Similarly, for the product $1_{hv\Gamma}1_{t\Gamma}$ to be non-zero we must have $hv\Gamma = t\Gamma$, i.e. $v\Gamma = h^{-1}t\Gamma$. Thus,

$$\begin{aligned} T_{g\Gamma,h\Gamma} * T_{h\Gamma,t\Gamma} &= 1_{g\Gamma} * 1_{hh^{-1}g\Gamma} * \Gamma (h^{-1}g)^{-1} h^{-1} t\Gamma * 1_{hh^{-1}t\Gamma} * 1_{t\Gamma} \\ &= 1_{g\Gamma} * \Gamma g^{-1} t\Gamma * 1_{t\Gamma} \\ &= T_{g\Gamma,t\Gamma} \,. \end{aligned}$$

Hence, $\{T_{g\Gamma,h\Gamma}\}_{g\Gamma,h\Gamma}$ is a set of matrix units. The fact that this set spans $C_c(G/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ follows readily from Theorem 6.1.14, noting that for $x \in G$ and $g\Gamma \in G/\Gamma$ we have

$$1_{x\Gamma} * \Gamma g\Gamma * 1_{xg\Gamma} = T_{x\Gamma,xg\Gamma} \,.$$

This finishes the proof. □

Theorem 10.1.2. *The full crossed product $C_0(G/\Gamma) \times_\alpha G/\Gamma$ exists and moreover*

$$C_0(G/\Gamma) \times_\alpha G/\Gamma \;\cong\; C_0(G/\Gamma) \times_{\alpha,r} G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma)) \,.$$

Denoting by $M : C_0(G/\Gamma) \to B(\ell^2(G))$ the *-representation by pointwise multiplication, we have that (M, ρ) is a covariant *-representation and $M \times \rho$ is a faithful *-representation of $C_0(G/\Gamma) \times_\alpha G/\Gamma$ with range $\mathcal{K}(\ell^2(G/\Gamma))$.

Proof: By Proposition 10.1.1 we have that $\{T_{g\Gamma,h\Gamma}\}_{g\Gamma,h\Gamma}$ is a set of matrix units that span $C_c(G/\Gamma) \times^{alg}_{\alpha} G/\Gamma$. Hence, the enveloping C^*-algebra of $C_c(G/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ must exist. As it is known, there exists only one C^*-algebra , up to isomorphism, generated by a set of matrix units indexed by G/Γ, and that is $\mathcal{K}(\ell^2(G/\Gamma))$. Hence, we necessarily have

$$C_0(G/\Gamma) \times_\alpha G/\Gamma \;\cong\; C_0(G/\Gamma) \times_{\alpha,r} G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma)) \,.$$

It has been shown in [19, Example 1.5] that (M,ρ) is a covariant pair, so that equality (10.1) holds. Since the action of G on itself is free, it follows readily from Proposition 6.4.3 that this means that (M,ρ) is a covariant *-representation.

Let us denote by $\phi : C_0(G) \to \mathbb{C}$ the *-representation given by evaluation at the identity element, i.e.

$$\phi(f) := f(e)\,,$$

and let $\widetilde{\phi}$ be its extension to $M(C_0(G)) \cong C_b(G)$. We claim that $\widetilde{\phi}_\alpha = M$, and this follows from the following computation, where $f \in C_c(G/\Gamma)$:

$$\begin{aligned} \widetilde{\phi}_\alpha(f)\delta_{h\Gamma} &= \widetilde{\phi}(\alpha_h(f))\delta_{h\Gamma} = \alpha_h(f)\,(h\Gamma h^{-1})\delta_{h\Gamma} \\ &= f(h\Gamma)\delta_{h\Gamma} = M(f)\delta_{h\Gamma}\,. \end{aligned}$$

Since M is faithful, it now follows from Theorem 8.3.6 that $M\times\rho$ is a faithful *-representation of $C_0(G/\Gamma)\times_{\alpha,r} G/\Gamma \cong C_0(G/\Gamma)\times_\alpha G/\Gamma$ in $B(\ell^2(G/\Gamma))$, whose image must necessarily be $\mathcal{K}(\ell^2(G))$. $\square$

As a corollary of our Stone-von-Neumann theorem we recover [19, Theorem 1.6] and we show that the covariant pre-*-representations of $C_c(G/\Gamma)\times_\alpha^{alg} G/\Gamma$ coincide with the covariant pairs of [19].

Corollary 10.1.3. *Let (G,Γ) be a Hecke pair, $\pi : C_0(G/\Gamma) \to B(\mathscr{H})$ a nondegenerate *-representation and $\mu : \mathcal{H}(G,\Gamma) \to L(\pi(C_c(G/\Gamma))\mathscr{H})$ a unital pre-*-representation. Then (π,μ) is a covariant pre-*-representation if and only if it is unitarily equivalent to an amplification $(1\otimes M, 1\otimes\rho)$ of (M,ρ). In particular we have*

- *i) All covariant pre-*-representation are covariant *-representations, and these are the same as the covariant pairs of [19].*
- *ii) A *-representation π is equivalent to an amplification of M if and only if there exists a *-representation μ of $\mathcal{H}(G,\Gamma)$ such that (π,μ) is a covariant *-representation.*

Proof: Let (π,μ) be a covariant pre-*-representation of $C_c(G/\Gamma)\times_\alpha^{alg} G/\Gamma$. Then its integrated form $\pi\times\mu$ extends to a nondegenerate *-representation of $C_0(G/\Gamma)\times_\alpha G/\Gamma$. By Theorem 10.1.2 $M\times\rho$ is a *-isomorphism between $C_0(G/\Gamma)\times_\alpha G/\Gamma$ and $\mathcal{K}(\ell^2(G/\Gamma))$, so that $(\pi\times\mu)\circ(M\times\rho)^{-1}$ is a nondegenerate *-representation of $\mathcal{K}(\ell^2(G/\Gamma))$. Since the algebra of compact operators has a trivial representation theory (see for example [37, Lemma B.34]) there exists a Hilbert space $\mathscr{H}$ such that $(\pi \times \mu) \circ (M \times \rho)^{-1}$ is unitarily equivalent to the representation $1 \otimes \mathrm{id}$ in $\mathscr{H} \otimes \ell^2(G/\Gamma)$. Hence, $(\pi \times \mu)$ is unitarily equivalent to $1 \otimes (M \times \rho)$. Now given the fact that (M, ρ) is a covariant *-representation, it is not difficult to see that $(1 \otimes M, 1 \otimes \rho)$ is also a covariant *-representation and moreover

$$1 \otimes (M \times \rho) = (1 \otimes M) \times (1 \otimes \rho)\,.$$

By Proposition 6.2.17 we then have that (π, μ) is unitarily equivalent to $(1 \otimes M, 1 \otimes \rho)$.

The converse is easier: suppose now that (π, μ) is equivalent ot an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ). Since $(1 \otimes M, 1 \otimes \rho)$ is a covariant *-representation, it follows that (π, μ) must also be a covariant *-representation.

Let us now check $i)$. As we have just proven, every covariant pre-*-representation is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ). Since, $(1 \otimes M, 1 \otimes \rho)$ is a covariant *-representation it follows that every covariant pre-*-representation is actually a covariant *-representation.

Let us now prove $ii)$. Suppose $\pi : C_0(G/\Gamma) \to B(\mathscr{H})$ is equivalent to an amplification of M, i.e. there exists a Hilbert space $\mathscr{H}_0$ and a unitary $U : \mathscr{H} \to \mathscr{H}_0 \otimes \ell^2(G/\Gamma)$ such that $\pi = U(1 \otimes M)U^*$. As $(U(1 \otimes M)U^*, U(1 \otimes \rho)U^*)$ is a covariant *-representation, we conclude that there exists a *-representation μ such that (π, μ) is a covariant *-representation. The converse follows easily from what we proved above: if there exists a *-representation μ of $\mathcal{H}(G, \Gamma)$ such that (π, μ) is a covariant *-representation, then (π, μ) is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ), and therefore π is unitarily equivalent to an amplification of M. $\square$

Chapter 11

Concluding remark: towards Katayama duality

The theory of crossed products by Hecke pairs developed in the previous chapters is intended for applications in non-abelian crossed product duality. We have already taken the first step in this direction, having established a Stone-von Neumann theorem for Hecke pairs which reflects the results of an Huef, Kaliszewski and Raeburn [19]. We believe that this theory of crossed products by Hecke pairs can be further applied and bring insight into the emerging theory of crossed products by coactions of homogeneous spaces ([10], [9]). The basic idea here is to obtain duality results for "actions" and "coactions" of homogeneous spaces (those coming from Hecke pairs).

In this chapter we will explain how our construction of a crossed product of a Hecke pair seems very suitable for obtaining a form of Katayama duality for homogeneous spaces arising from Hecke pairs, with respect to what we call the *Echterhoff-Quigg crossed product*.

Let δ be a coaction of a discrete group G on a C^*-algebra B and $B \times_\delta G$ the corresponding crossed product. We follow the conventions and notation of [10] for coactions and their crossed products. As it is known, there is an action $\widehat{\delta}$ of G on $B \times_\delta G$, called the *dual action*, determined by

$$\widehat{\delta}_s\big(j_B(a)j_G(f)\big) := j_B(a)j_G(\sigma_s(f))\,, \qquad \forall a \in B, f \in C_0(G), s \in G\,,$$

where σ denotes the action of right translation on $C_0(G)$, i.e. $\sigma_s(f)(t) := f(ts)$.

Katayama's duality theorem (original version comes from [25, Theorem 8]) is an analogue for coactions of the duality theorem of Imai and Takai. A general version of it states that we have a canonical isomorphism

$$(B \times_\delta G) \times_{\widehat{\delta},\omega} G \cong B \otimes \mathcal{K}(\ell^2(G))\,, \tag{11.1}$$

for some C^*-completion of the *-algebraic crossed product $(B \times_\delta G) \times^{alg}_{\widehat{\delta}} G$. This C^*-completion $(B \times_\delta G) \times_{\widehat{\delta},\omega} G$ lies in between the full and the reduced crossed products, and the coaction δ is called *maximal* (respectively, *normal*) if this C^*-crossed product is the full (respectively, the reduced) crossed product.

We would like to extend this duality result for coactions of homogeneous spaces G/Γ. In spirit we should obtain an isomorphism of the type

$$(B \times_\delta G/\Gamma) \times_{\widehat{\delta},\omega} G/\Gamma \;\cong\; B \otimes \mathcal{K}(\ell^2(G/\Gamma))\,. \tag{11.2}$$

Of course, the expression on the left hand side makes no sense unless Γ is normal in G (in which case, the above is just Katayama's result), and there are a few reasons for that. First, it does not make sense in general for a homogeneous space to coact on a C^*-algebra, which consequently makes it difficult to give meaning to $B \times_\delta G/\Gamma$. Secondly, it also does not make sense in general for a homogeneous space G/Γ to act (namely, by $\widehat{\delta}$) on a C^*-algebra.

The second objection can be overcomed by simply using our definition of a crossed product by (an "action" of) a Hecke pair (G, Γ). The first objection can be overcomed because, even though there is no definition of a coaction of a homogeneous space, it is possible to define C^*-algebras $B \times_\delta G/\Gamma$ which can be thought of as crossed products of B by a coaction of G/Γ ([10], [9]). In this way the iterated crossed product in expression (11.2) may have a true meaning. This is the approach we suggest towards a generalization of Katayama's result.

It is our point of view that such a Katayama duality result can hold when $B \times_\delta G/\Gamma$ is a certain C^*-completion of the algebra $C_c(\mathcal{B} \times G/\Gamma)$ defined by Echterhoff and Quigg in [10], which we dub the *Echterhoff and Quigg's crossed product*, a terminology already used by an Huef, Kaliszewski and Raeburn in [19] for $C^*(\mathcal{B} \times G/\Gamma)$ in case we start with a maximal coaction of G on B. Hence, we have to ensure that Echterhoff and Quigg's algebra $C_c(\mathcal{B} \times G/\Gamma)$ falls in our set up for defining crossed products by Hecke pairs, and that is what we explain now.

We recall briefly the construction of Echterhoff and Quigg, and the reader is advised to read our Example 5.2.4 again. We start with a coaction δ of a discrete group G on a C^*-algebra B, and we denote by $\mathcal{B}$ its associated Fell bundle. Following [11, Section 3] we denote by $\mathcal{B} \times G$ the corresponding Fell bundle over the groupoid $G \times G$, whose fibers are

$$(\mathcal{B} \times G)_{(s,t)} := \mathcal{B}_s\,.$$

We recall that the multiplication and inversion in $G \times G$ are given by

$$(s, tr)(t, r) = (st, r) \qquad \text{and} \qquad (s, t)^{-1} = (s^{-1}, st)\,.$$

An important property of $C_c(\mathcal{B} \times G/\Gamma)$ is that it embeds densely in the coaction crossed product $B \times_\delta G$, by identifying (a_s, t) with $j_B(a)j_G(1_t)$. In this setting we have that $B \times_\delta G \cong C^*(\mathcal{B} \times G) \cong C_r^*(\mathcal{B} \times G)$, as stated in [11, Corollary 3.4].

The dual action $\widehat{\delta}$ of G on $B \times_\delta G$ is determined by $\widehat{\delta}_g(j_B(a)j_G(1_t)) = j_B(a)j_G(1_{tg^{-1}})$, which on the generators of $C_c(\mathcal{A} \times G)$ means

$$\widehat{\delta}_g(a_s, t) := (a_s, tg^{-1}) .$$

Let $H \subseteq G$ be a subgroup and $\mathcal{B} \times G/H$ the corresponding Fell bundle over the groupoid $G \times G/H$, as in [10], whose operations are defined by

$$(s, trH)(t, rH) = (st, rH) \qquad \text{and} \qquad (s, tH)^{-1} = (s^{-1}, stH) .$$

The Echterhoff and Quigg algebra is just the algebra $C_c(\mathcal{B} \times G/H)$ of finitely supported section of this Fell bundle.

Let us now consider the case of a Hecke pair (G, Γ) to see that the conditions of our definition of crossed products by Hecke pairs are met, and see that it makes sense to define $C_c(\mathcal{B} \times G/\Gamma) \times_{\widehat{\delta}}^{alg} G/\Gamma$.

For this we take X as the groupoid $G \times G$. This groupoid carries a natural right G-action given by

$$(s, t)g := (s, tg) . \tag{11.3}$$

Since this action is free, it is H-good and satisfies the H-intersection property for any subgroup $H \subseteq G$. We take the Fell bundle $\mathcal{A}$ over X as the bundle $\mathcal{B} \times G$, which clearly has G-invariant fibers. Moreover, we see that the dual action $\widehat{\delta}$ on $C_c(\mathcal{A}) = C_c(\mathcal{B} \times G)$ is just the action which is induced from the action (11.3) of G on X (via Proposition 5.1.5).

Now we observe that, for any subgroup $H \subseteq G$, the orbit groupoid X/H is canonically identified with the groupoid $G \times G/H$, simply by $(s, t)H \mapsto (s, tH)$. This canonical identification is easily seen to be a groupoid isomorphism, so that X/H and $G \times G/H$ are "the same" groupoid. Under this identification, the Fell bundle $\mathcal{A}/H$ is just the Fell bundle $\mathcal{B} \times G/H$, and therefore we can canonically identify $C_c(\mathcal{A}/H)$ with $C_c(\mathcal{B} \times G/H)$.

We therefore conclude that we can define the *-algebraic crossed product $C_c(\mathcal{B} \times G/\Gamma) \times_{\widehat{\delta}}^{alg} G/\Gamma$. We expect that there is a C^*-completion of the Echterhoff and Quigg algebra $C_c(\mathcal{B} \times G/\Gamma)$, which we would like to call *the* Echterhoff and Quigg's crossed product, for which a form of Katayama duality as in (11.2) holds.

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